55. *Remarks on Torus Knots*

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As is well known the knot group of a torus knot is presented as the group generated by two elements, a, b, with a single defining relation $a^p=b^q$, p, q being integers, $(p,q)=1$. Conversely, if a knot group is presented as above, is the knot necessarily a torus knot? If our consideration is restricted to alternating knots, this problem can easily be answered in the affirmative. In fact we have the following

[Theorem] The knot group G of an alternating knot is presented as follows:

$$G_{p,q}=\{a,b:a^p\cdot b^{-q}\},$$

if and only if it is a torus knot.

To prove this theorem the following lemma which is the group theoretic formulation of the main theorems in [1], is basic.

[Lemma 1] The knot group G of any alternating knot is presented as the free product of two free groups A, B of the same ranks, m, with an amalgamated subgroup H, a free group of a rank $2m-1$, where $m\geqq 1$.

Since A, B are free groups, we immediately have [3]

[Lemma 2] The center C of G is a free group.

Thus the rank of C is a knot invariant. However, it is trivial for all alternating knots except torus knot. In fact:

[Lemma 3] The center of G is trivial if the rank m of A or B is greater than one. If $m=1$, then the rank of the center of G is one.

Now given an alternating knot, we can obtain the group presentation as indicated in Lemma 1, by using the alternating knot projection. In this presentation, we know that $m=1$ if and only if it is a torus knot [2]. Then Lemma 3 follows that the knot groups of any alternating knots except torus knots can not be presented as $G_{p,q}$. This completes the proof of this theorem.

References

[1] R. J. Aumann: Asphericity of alternating knots, Ann. of Math., **64**, 374–392 (1956).
[2] K. Murasugi: On alternating knots, Osaka Math. J., **12**, 277–303 (1960).
[3] B. H. Neumann: An essay on free products of groups with amalgamations, Phil. Trans. Royal Soc. London, **246**, 503–554 (1954).