48. On the Definition of the Cross and Whitehead Products in the Axiomatic Homotopy Theory. II

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(Comm. by K. KUNUGI, M.J.A., April 12, 1961)

1. Introduction. In the preceding paper I, we have described the definition of the cross and Whitehead products. In this paper we shall show a few properties of the cross and Whitehead products as consequences of their definition and prove the existence and uniqueness of these products.

2. Immediate consequences from the axiom. Consider two maps $f:(X, x_0)\to(X', x'_0)$ and $g:(Y, y_0)\to(Y', y'_0)$ and let $f\times g:(X\times Y, X\vee Y, (x_0, y_0))\to(X'\times Y', X'\vee Y', (x'_0, y'_0))$ be a map defined by $(f\times g)(x, y)=(f(x), g(y))$.

Proposition 1. *For* $\alpha\in\pi_m(X, x_0)$, $\beta\in\pi_n(Y, y_0)$, *we have*

$$(f\times g)_{\#}(\alpha\times\beta)=f_{\#}\alpha\times g_{\#}\beta.$$

This is easily proved and the proof is omitted.

Now let $\tau: X\times Y\to Y\times X$ be a map such that $\tau(x, y)=(y, x)$.

Proposition 2. *For* $\alpha\in\pi_m(X, x_0)$, $\beta\in\pi_n(Y, y_0)$, *we have*

$$\tau_{\#}(\alpha\times\beta)=(-1)^{mn}(\beta\times\alpha). \tag{1}$$

In order to prove this, we shall need the following lemma, whose proof will be omitted.

Lemma 3. *Let* $f, g:(X, x_0)\to(Y, y_0)$ *be H-homomorphisms between H-spaces* X *and* Y *with units* x_0, y_0 *respectively. An H-homomorphism* $h=f\cdot g:(X, x_0)\to(Y, y_0)$ *is defined by* $h(x)=f(x)\cdot g(x)$, $x\in X$. *Then we have* $h_{\#}(\alpha)=f_{\#}(\alpha)+g_{\#}(\alpha)$, *for* $\alpha\in\pi_n(X, x_0)$, $n>0$. *If* X *and* Y *are loop spaces,* $\pi_0(X, x_0)$ *and* $\pi_0(Y, y_0)$ *may be considered as groups. In this case the above relation holds also.*

Proof of Prop. 2. In cases $m=n=0$; $m=0$, $n>0$; $m>0$, $n=0$, we can show directly by definition that the formula (1) holds. Now we assume that the formula (1) holds for $k<m$, $l<n$. Let $\Omega\tau: \Omega X\times\Omega Y\to\Omega Y\times\Omega X$, $\tau': X\vee Y\to Y\vee X$ and $\Omega\tau': \Omega(X\vee Y)\to\Omega(Y\vee X)$ be maps induced by τ. Then $\Omega\partial\tau_{\#}(\alpha\times\beta)=\Omega\tau'_{\#}\partial(\alpha\times\beta)=(\Omega\tau')_{\#}\Omega\partial(\alpha\times\beta)=(-1)^{n-1}(\Omega\tau')_{\#}\varphi_{\natural}(\Omega\alpha\times\Omega\beta)=(-1)^{n-1}((\Omega\tau')\circ\varphi)_{\natural}(\Omega\alpha\times\Omega\beta)$. A map $(\Omega\tau')\circ\varphi: \Omega X\times\Omega Y\to\Omega(Y\vee X)$ is defined by $((\Omega\tau')\circ\varphi)(x, y)=(y_0, x)(y, x_0)(y_0, x^{-1})(y^{-1}, x_0)$. On the other hand, $(\varphi\circ(\Omega\tau))(x, y)=(y, x_0)(y_0, x)(y^{-1}, x_0)(y_0, x^{-1})$. Therefore $(\Omega\tau')\circ\varphi=(\varphi\circ(\Omega\tau))^{-1}$. By Lemma 3, we have $((\Omega\tau')\circ\varphi)_{\natural}=-(\varphi\circ(\Omega\tau))_{\natural}$. Hence $\Omega\partial\tau_{\#}(\alpha\times\beta)=(-1)^n(\varphi\circ(\Omega\tau))_{\natural}(\Omega\alpha\times\Omega\beta)=(-1)^n(\varphi_{\natural}\circ(\Omega\tau)_{\#})(\Omega\alpha\times\Omega\beta)=(-1)^n\varphi_{\natural}((-1)^{(m-1)(n-1)}(\Omega\beta\times\Omega\alpha))=(-1)^{mn}\Omega\partial(\beta\times\alpha)$. Thus we have $\tau_{\#}(\alpha\times\beta)=(-1)^{mn}\beta\times\alpha$.