60. On the Example of an Inhomogeneous Partial Differential Equation without Distribution Solutions

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1. Let Ω be a domain in Euclidean n-space and let P be a partial differential operator with constant coefficients.

Here we consider the distribution equation

$$PS=F \tag{1}$$

where F is a given distribution in $D'(\Omega)$ and S is the solution in $D'(\Omega)$. It was shown by B. Malgrange [1] that for elliptic operators P and for any domain Ω there is a solution S of (1) and that for hypoelliptic operators P the existence theorem is always valid whenever Ω is a P-convex domain, i.e. to every compact set $K\subset\Omega$ there exists another compact set $K'\subset\Omega$ such that supp $\varphi\subset K'$ for every $\varphi\subset\mathfrak{D}(\Omega)$ such that supp $P'\varphi\subset K$. Furthermore it is easily shown applying usual methods used by several authors that for any geometrically convex domain Ω and for any P the existence theorem is also valid.

In the present note I shall show using a result of F. John's that for the hyperbolic operator the existence theorem is not true for some P-convex domain.

2. To show a counter example we use the following

Lemma 1. Let Ω_i be a bounded subdomain of a domain $\Omega\subset R_n$ such that

$$\Omega\supset\bar{\Omega}_i\not\supset\Omega_{i+1} \qquad \text{for any integer } i=1,2,\cdots,$$

and let K_i be $(n-1)$-dim surfaces such that

$$K_i\subset\Omega_i-\bar{\Omega}_{i-1},$$

$$K_i\to\text{a part of the boundary } \dot{\Omega} \text{ of } \Omega.$$

Furthermore we assume that for some increasing sequence $\{s_i\}$ of integers there exist functions f_i such that

$$\text{supp } f_i\subset\Omega_i \tag{2}$$

$$f_i\in C^{s_i-1}(\Omega_i), \tag{3}$$

$$f_i\in C^{s_i}(U(K_i)-K_i), \tag{4}$$

but for some D^{s_i},

$$D^{s_i}f_i\notin L_{p_i}(U(K_i)) \qquad (\infty>p_i>1), \tag{5}$$

where $U(K_i)$ is an open set, and

$$P'f_i\in C^{s_i-p}(\Omega_i)\frown C^\infty(\Omega_i-K) \tag{6}$$

where p is the degree of P and K is a fixed compact subset of Ω.

Then there exists a distribution F such that (1) has no distribution solutions for Ω.