110. On the Distribution of the Spectra of Normal Operators in Hilbert Spaces

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(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1961)

We shall define in advance the symbols, which will be used in this paper, as follows:

Definition. Let $\mathfrak{H}$ be the complex abstract Hilbert space which is complete, separable and infinite dimensional; let N be a normal operator in $\mathfrak{H}$; let $\rho(N), \sigma(N)=\{z_\nu\}_{\nu=1,2\cdots}$ and $\Delta(N)$ be the resolvent set, the point spectrum and the continuous spectrum of N respectively; let $\{K(z)\}$ be the complex spectral family associated with N; let K_ν be the eigenprojector of N corresponding to the eigenvalue z_ν; and let 0_0 and 0_e be the null operator and the null element in $\mathfrak{H}$ respectively.

We now suppose that λ_0 belongs to $\Delta(N)$ but not to the set of accumulation points of $\sigma(N)$. Then, by applying the factorization of $K(z)$ by the spectral families of the self-adjoint operators $\frac{1}{2}(N+N^*)$ and $\frac{1}{2i}(N-N^*)$ on $\mathfrak{D}(N)$, we can first verify that λ_0 is not an isolated point of $\Delta(N)$. If we next denote by $\Delta_{\varepsilon,\lambda_0}$ the intersection of $\Delta(N)$ and a suitably small ε-neighborhood of λ_0, then, by the application of this result and the fact that $\rho(N)$ is an open set, we can find that the points of $\Delta_{\varepsilon,\lambda_0}$ are continuously distributed. In addition, there is no difficulty in showing that the dimension of $K(\Delta_{\varepsilon,\lambda_0})\mathfrak{H}$ is denumerably infinite, however small $\varepsilon>0$ may be. After these preliminaries, we shall turn to our purpose.

Theorem 1. Let D be a domain in the complex λ-plane whose boundary ∂D is a rectifiable closed Jordan curve. If the closure $\overline{D}$ of D is a subset of the resolvent set $\rho(N)$ of a normal operator N in $\mathfrak{H}$, then

$$(1) \qquad \int_{\partial D}(\lambda I-N)^{-1}d\lambda=0_0,$$

where the curvilinear integration is taken in the counterclockwise direction; and if, conversely, (1) holds, D is a subset of $\rho(N)$.

Proof. We now divide ∂D into n pieces by points $\lambda_1, \lambda_2, \cdots, \lambda_n$ on itself and let $|\lambda_{\alpha+1}-\lambda_\alpha|\to 0$, $(\alpha=1, 2, \cdots, n; \lambda_{n+1}=\lambda_1)$, by allowing n to become infinite. Then, remembering the facts that $\int_{\partial D}\frac{d\lambda}{\lambda-z}=0$ or $2\pi i$, according as z lies outside or inside ∂D, and that $\rho(N)$ is an