100. On Some Measure-Theoretic Results in Curve Geometry

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1. Preliminary remarks. Let $\boldsymbol{R}^m$ be a Euclidean space of any dimension $m \geqq 1$, where we identify $\boldsymbol{R}^1$ with the real line $\boldsymbol{R}$. By a *curve* (in $\boldsymbol{R}^m$) we shall understand any mapping of $\boldsymbol{R}$ into $\boldsymbol{R}^m$. Thus a curve is no other than a real function when $m=1$. The letter φ will be reserved for a given curve throughout what follows. Let us add that all the sets (and intervals) considered will be situated in $\boldsymbol{R}$ unless stated otherwise or another meaning is obvious from the context. For each set E we define the *length* and the *measure-length* of φ on E as in [1]§37 and in [2]§4 respectively. The former will be denoted by $L(\varphi; E)$ as before, but the latter by $L_*(\varphi; E)$ in this note.

As a matter of fact, the measure-length $L_*(\varphi; E)$ depends not only on the behaviour of φ within the set E but also on its definition for the points outside E, even the case where φ is continuous being no exception to this observation. So long as we are concerned with locally rectifiable curves, however, this does not cause any serious obstacle to the construction of a reasonable theory of measure-length. It is when we step forward beyond such curves that things begin to show themselves unfavourable to us. One way of avoiding the difficulty that thus arises is to abandon the study of the measure-length by itself and to direct our chief interest to certain other set-functions (to be defined in §2) which serve as substitutes for the measure-length and whose values for any set E depend solely on the behaviour of the curve within E. Some of their fundamental properties will constitute the subject matter of the present note.

2. Reduced and Hausdorff measure-lengths of a curve. Given a set E, let us consider an arbitrary sequence (finite or infinite) of its subsets, $\Delta=\langle E_1, E_2, \cdots\rangle$, such that $[\Delta]=E_1 \smile E_2 \smile \cdots = E$. The infimum, for all Δ, of the sum $L(\varphi; \Delta)=L(\varphi; E_1)+L(\varphi; E_2)+\cdots$ will be termed *reduced measure-length* of the curve φ over the set E and denoted by the symbol $\Xi(\varphi; E)$. Let us now write Δ_ε for Δ when especially every E_n has its diameter $\mathrm{d}(E_n)$ smaller than a positive number ε, the diameter of the void set being understood to be zero. Consider the images $\varphi[E_n]$ of the sets E_n under the mapping φ and denote by $\Gamma_\varepsilon(\varphi; E)$ the infimum, for all Δ_ε, of the sum $\mathrm{d}(\varphi[E_1])+\mathrm{d}(\varphi[E_2])+\cdots$. When $\varepsilon \to 0$, this infimum plainly tends in