141. On the Uniform Distribution of Sequences of Integers

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1. Introduction. Consider an infinite sequence $A=(a_n)$ of integers. For any integers j and $m\geqq 2$ we denote by $A(N,j,m)$ the number of terms a_n $(1\leqq n\leqq N)$ satisfying the condition $a_n\equiv j \pmod{m}$. The sequence A is said to be *uniformly distributed modulo* m if the limit

$$\lim_{N\to\infty}\frac{1}{N}A(N,j,m)=\frac{1}{m}$$

exists for all j, $1\leqq j\leqq m$. If A is uniformly distributed modulo m for every integer $m\geqq 2$, we say simply that A is *uniformly distributed.*

I. Niven [1] has exhibited a number of interesting properties of uniformly distributed sequences of integers. Among others he proved that the sequence A, defined by $a_n=[ns]$, is uniformly distributed if and only if s is irrational or $s=1/k$ for some non-zero integer k, and that the uniform distribution of the sequence $([ns])$ for every irrational s is equivalent to the well-known theorem that the sequence of the fractional parts of ns is uniformly distributed modulo 1 for every irrational s (cf. e.g. [2]). It is not difficult to show that, for every infinite sequence (a_n) of mutually distinct integers, the sequence $([a_n s])$ is uniformly distributed for almost all real numbers s. (Here 'almost all' means 'all but a set of Lebesgue measure zero'.)

The main purpose of the present note is to obtain some criteria for sequences of integers to be uniformly distributed (with or without the reference to modulus m).

Let us put, for brevity's sake,

$$e(x)=\exp(2\pi ix).$$

We shall prove:

Theorem 1. *Let $A=(a_n)$ be an infinite sequence of integers. A necessary and sufficient condition that A be uniformly distributed modulo m, where $m\geqq 2$, is that*

$$(1)\qquad \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e\left(a_n\frac{h}{m}\right)=0$$

for all $h=1,2,\cdots,m-1$.

Hence:

Corollary. *A necessary and sufficient condition that an infinite sequence $A=(a_n)$ of integers be uniformly distributed is that*