# 2. *Some Characterizations of Fourier Transforms. III*

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1. In this paper we shall denote with $\mathfrak{P}$ the space of all functions on the real number field of class $C^\infty$ whose derivatives decrease rapidly and with $\mathfrak{D}$ the subspace of $\mathfrak{P}$ consisting of all functions in $\mathfrak{P}$ with compact support. For the topology $\mathfrak{P}$ and $\mathfrak{D}$ see the Schwartz's book ([4]). And we denote $\varphi(x+h)$ with $\varphi_h(x)$ as a function of $x$. The purpose of this paper is to prove the following

Theorem. *Let $T$ be a continuous linear mapping from $\mathfrak{P}$ to itself which satisfies the following conditions*:

I) $T^2\varphi(x)=\varphi(-x)$,

II) $T(\varphi*\psi)=T\varphi\cdot T\psi$.

*Then $T\varphi(x)$ must be equal to $E\varphi(x)$ or $E\varphi(-x)$, where $E\varphi(x)$ is the Fourier transform $\int_{-\infty}^{\infty} e^{2\pi ixt}\varphi(t)\,dt$ of $\varphi(x)$.*

2. First we shall prove a few lemmas.

Lemma 1. *Let $\varphi$, $\psi$ be elements of $\mathfrak{D}$ and the support of $\varphi$ be contained in $[a,b]$. If we put*

$$f_n(x)=\frac{b-a}{n}\sum_{j=1}^{n}\varphi(x-h_j)\psi(h_j)$$

*for every natural number $n$, where $h_j=a+\frac{(b-a)j}{n}$, then the series $f_1(x), f_2(x),\cdots$ converges to $\varphi*\psi$ in $\mathfrak{D}$ and,* a fortiori, *in $\mathfrak{P}$.*

We omit the proof of this lemma because it is very easy.

Lemma 2. *There is a continuous function $r(x)$ on the real number field such that*

$$T\varphi_h(x)=\exp(2\pi ihr(x))T\varphi(x)$$

*for every function $\varphi$ in $\mathfrak{P}$ and every couple of real numbers $h$ and $x$.*

Proof. For any given $x$ there exists an element $\psi$ of $\mathfrak{P}$ such that $T\psi(x)\neq 0$ by Condition I. Let us denote $\frac{T\psi_h(x)}{T\psi(x)}$ with $u(h,x)$ or $u(h)$. Because

$$(\varphi*\psi)_h=\varphi_h*\psi=\varphi*\psi_h$$

we get

$$T\varphi_h(x)T\psi(x)=T\varphi(x)T\psi_h(x)$$

by Condition II. Therefore

$$T\varphi_h(x)=T\varphi(x)u(h)$$

for every $\varphi$ in $\mathfrak{P}$. From this we can claim $u(h)\neq 0$, because there exists an element $\varphi$ of $\mathfrak{P}$ such that $T\varphi_h(x)\neq 0$. Also we see that if