32. *Further Results in Lebesgue Geometry of Curves*

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1. Proof of a theorem. As heretofore we shall be concerned with curves situated in a Euclidean space $\boldsymbol{R}^m$ of dimension $m\geqq 2$. Sets, by themselves, will always mean sets of real numbers unless specified to the contrary. To prove the theorem stated at the end of [4], we shall begin with a lemma in which the points of $\boldsymbol{R}^m$ will be called vectors for convenience.

LEMMA. (i) *We have $(x\diamond y)|x|<4|x-y|$ for every distinct pair of nonvanishing vectors x and y.* (ii) *Given a positive number $\varepsilon\leqq 1/2$ and four vectors p, q, p', q' such that $p\neq 0, q\neq 0$, and $p\diamond q\neq 0$, write for short $\theta=(p\diamond q)/4$ and suppose that*

$$|p'-p|\leqq\varepsilon\theta|p|,\quad |q'-q|\leqq\varepsilon\theta|q|.$$

Then the two vectors $p-q$ and $p'-q'$ are nonvanishing and the angle between them is less than 8ε.

PROOF. *re* (i): The identity $|x-y|^2=|x|^2+|y|^2-2|x|\cdot|y|\cos\alpha$, where $\alpha=x\diamond y$, implies that if $\alpha>\pi/2$, then $4|x-y|>4|x|>\alpha|x|$. On the other hand we always have $|x-y|\geqq|x|\sin\alpha$ on account of the identity $|x-y|^2-(|x|\sin\alpha)^2=(|x|\cos\alpha-|y|)^2$. When $\alpha\leqq\pi/2$, we therefore find, in view of the well-known inequality $\pi\sin\alpha\geqq 2\alpha$, that $\alpha|x|\leqq 2|x|\sin\alpha\leqq 2|x-y|$. This establishes (i).

re (ii): Write $w=p-q$ and $w'=p'-q'$, so that $w\neq 0$ since $p\diamond q\neq 0$. Part (i) proved already implies $\theta|p|<|w|$ and $\theta|q|<|w|$. Hence

$$|p'-p|+|q'-q|\leqq\varepsilon\theta|p|+\varepsilon\theta|q|<2\varepsilon|w|.$$

This, united with the evident relation $|w|\leqq|w'|+|p'-p|+|q'-q|$, gives $|w'|>(1-2\varepsilon)|w|\geqq 0$, so that w' cannot vanish. Putting now for brevity $\lambda=(w\diamond w')/4$ and using (i) again, we find further

$$\lambda|w|\leqq|w-w'|\leqq|p'-p|+|q'-q|<2\varepsilon|w|.$$

Since $w\neq 0$, it follows that $\lambda<2\varepsilon$, Q. E. D.

THEOREM. *A light curve φ is spherically representable on both sides provided that it is locally straightenable.*

PROOF. We can associate with each point $a\in\boldsymbol{R}$ a positive number δ (depending on a) such that $\varphi(t)\neq\varphi(a)$ whenever $a<t\leqq a+\delta$. For otherwise there would exist a strictly decreasing sequence of points $t_1>t_2>\cdots$ tending to a and such that $\varphi(t_n)=\varphi(a)$ for each $n=1,2,\cdots$. Consider now the interval $K_n=[t_{n+1}, t_n]$ for each n. Then the curve φ, which is light by hypothesis, could not be constant on K_n, so that $\Omega(\varphi; K_n)\geqq\pi$ on account of [1]§60. In view of superadditivity of