44. On Complex Dirichlet Principle

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1. Let R, R' be a couple of compact Riemann surfaces with the same positive genus. Consider a conformal metric $\eta = ds_{\mathfrak{q}}^2 = \rho(w)|dw|^2$ introduced on R'; $\rho(w)|dw|^2$ remains invariant under any conformal transformations of the local parameter w attached to the point $\mathfrak{q} \in R'$ in question; $\rho(w)$ shall be positive and continuous in w.

Given a smooth homeomorphism f, mapping R onto R', we set

$$\begin{aligned} ds_{\mathfrak{q}}^2 &= E_f dx^2 + 2F_f dxdy + G_f dy^2 \\ &= (E_f + G_f)|dz|^2/2 + \mathrm{Re}\,\{(E_f - G_f - 2iF_f)dz^2\}/2, \end{aligned}$$

where $z = x + iy$ is a local parameter near the point $\mathfrak{p} = f^{-1}(\mathfrak{q})$. Then

$$I[f] = \frac{1}{2}\iint_R (E_f + G_f)dxdy$$

is regarded as a functional in f for the fixed η. In the brief but perspicacious paper due to Gerstenhaber-Rauch [2], one will find the following

DEFINITION. *f is called harmonic relative to η, when the quadratic differential $(E_f - G_f - 2iF_f)dz^2$ is holomorphic on R;*[1)]
and

THEOREM. *If $I[f] \leq I[g]$ holds for any topological mapping g from R to R' homotopic to f, f is harmonic relative to η.*

The content is, however, extremely heuristic and their reasoning involves somewhat essential gaps unfortunately in the management of the extremum problem: no mention was made of an admissibility condition for concurrence maps. It will be desirable to treat this problem from the purely variational point of view, which has motivated to write the present short note. Roughly speaking, variational problems have two demands which seem to be mutually exclusive in appearance; firstly, the *argument* that extremizes the given functional must be again admitted to concurrence (compactness); secondly, all the arguments in a suitable "neighbourhood" of the extremal must be admitted to concurrence (interiority). Therefore our task is to specify the family of admissible maps for this extremum problem.

The class of quasi-conformal mappings brings hardly any advantages to this purpose; the set of all quasi-conformal maps is not

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1) If, locally considered, $w(z) \in C^2$ and η is intrinsic metric, the holomorphy of $E - G - 2iF$ implies $\Delta w = 0$ at every point where the Jacobian $|\partial w/\partial z|^2 - |\partial w/\partial \bar{z}|^2$ does not vanish.