94. *Dirac Space*

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As is well-known, Dirac [1] gave a very elegant foundation of quantum mechanics, which contained however some self-contradictory concepts from the mathematical point of view. J. von Neumann [5] gave another foundation of the theory based upon his spectral theory of self-adjoint operators of Hilbert space. However the whole spectral theory is in fact not necessary and what Dirac actually needs is only that the self-adjoint operators $t\cdot$ and $\frac{1}{i}\frac{d}{dt}$ could be put in diagonal forms. There is also another justification of "improper" functions introduced by Dirac to put these operators in the diagonal forms by means of the theory of distributions of L. Schwartz. But this theory is not adequate to interpret inner product used in Dirac's theory. The purpose of this paper is to show that we can interpret the theory of Dirac in a more natural and mathematically rigorous way.

Recently A. Robinson developed non standard analysis [6], which is an adequate non Archimedian extension of real number field, and in which he succeeded to define infinite and infinitesimal and to develop rigorously infinitesimal calculus of Leibniz and Euler.

In this paper, we shall define Dirac space by an ultraproduct of $L_2(-\infty, \infty)$ and justify Dirac's method by using Robinson's consideration of infinite and infinitesimal.

Let I be the set of all positive integers.
A family $\mathfrak{F}$ of non-empty subsets of I is called a filter over I if

(i) $F_1 \in \mathfrak{F}$ and $F_2 \in \mathfrak{F}$ imply $F_1 \frown F_2 \in \mathfrak{F}$, and

(ii) $F_1 \in \mathfrak{F}$ and $F_1 \subseteq F_2 \subseteq I$ imply $F_2 \in \mathfrak{F}$.

A filter over I is an ultrafilter if it is maximal among the class of filters over I. It is easily proved that a necessary and sufficient condition that $\mathfrak{F}$ be an ultrafilter over I is that for $A \subseteq I$, $A \in \mathfrak{F}$ if and only if $I-A \notin \mathfrak{F}$. Hereafter we shall fix an ultrafilter $\mathfrak{F}_0$ over I which does not contain any finite subsets of I.

Now let R_0 be the set of all real numbers. Let a and b be elements of R_0^I. We use the notation $a=(a_1, a_2, \cdots)$ and $b=(b_1, b_2, \cdots)$ as usual, where a_i and b_i are real numbers and called the i-th coordinates of a and b respectively. $a \equiv b$, $a<b$, $a+b$ and $a\cdot b$ are defined to be $\{i \mid a_i = b_i\} \in \mathfrak{F}_0$, $\{i \mid a_i < b_i\} \in \mathfrak{F}_0$, $(a_1+b_1, a_2+b_2, \cdots)$ and $(a_1\cdot b_1, a_2\cdot b_2, \cdots)$ respectively. $\equiv$ is a congruence relation compatible with $<$, $+$, and $\cdot$.