135. A Proof of Kotaké and Narasimhan's Theorem

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(Comm. by Z. SUETUNA, M.J.A., Nov. 12, 1962)

We shall give a simple proof of the following theorem announced by Kotaké and Narasimhan [1].

Theorem. *Let $P=P(x,D)$ be a linear elliptic differential operator of order m with analytic coefficients in a domain $\Omega\subset R^n$. Then, a function $u=u(x)$ is analytic in Ω if and only if it satisfies*

$$(1)\qquad \|P^p u\|_{L^2(G)}\leqq B^{p+1}(pm)! \qquad (p=0,1,2,\cdots)$$

for every relatively compact subdomain $G\subset\Omega$ with a constant B depending only P, G and u.

Proof of Sufficiency. u is in $C^{pm-[n+1]/2}(\Omega)$ if $P^p u$ is in $L^2_{\mathrm{loc}}(\Omega)$. Therefore we may suppose that u is infinitely differentiable.

For functions f in $C^\infty(G)$ we define

$$\|\nabla^q f\|_\delta=\sum_{|\alpha|=q}\|D^\alpha f\|_{L^2(G_\delta)},$$

where G_δ is the set of points $x\in G$ such that the distance from x to the boundary of G is larger than δ. We shall make use of the following apriori inequalities (see [3] for a proof).

$$(2)\qquad \|\nabla^m f\|_{\delta+\sigma}\leqq C(\|Pf\|_\sigma+\delta^{-m}\|f\|_\sigma),$$

$$(3)\qquad \|\nabla^{m-r}f\|_{\delta+\sigma}\leqq C\varepsilon^r(\|\nabla^m f\|_\sigma+(\delta^{-m}+\varepsilon^{-m})\|f\|_\sigma)\qquad (0\leqq r\leqq m).$$

ε may take an arbitrary positive number and the constant C depends only on P and G.

We fix a positive constant ρ and define the semi-norm $N^{pm}(u)$ by

$$N^{pm}(u)=\sup_{\delta\leqq\rho}\delta^{pm}\|\nabla^{pm}u\|_\delta.$$

First we shall prove that if ρ is sufficiently small, then

$$(4)\qquad N^{pm}(u)\leqq C_0\left\{N^{(p-1)m}(Pu)+\sum_{q=0}^{p-1}\frac{(pm)!}{(qm)!}N^{qm}(u)\right\}$$

holds for every $u\in C^\infty(G)$ with a constant C_0 independent of u and $p=1,2,\cdots$.

When $p=1$, (4) is obviously valid with $C_0=2^m C$. In case $p+1\geqq 2$, it follows from (2) that

$$N^{(p+1)m}(u)=\sup_{(p+2)\delta\leqq\rho}((p+2)\delta)^{(p+1)m}\|\nabla^{(p+1)m}u\|_{(p+2)\delta}$$
$$\leqq 9^m C\sup_{(p+2)\delta\leqq\rho}(p\delta)^{(p+1)m}\{\|P\nabla^{pm}u\|_{(p+1)\delta}+\delta^{-m}\|\nabla^{pm}u\|_{(p+1)\delta}\}.$$

Because of the analyticity of the coefficients of $P(x,D)$, their r-th derivatives are majorated by $A^{r+1}r!$ with a constant $A\geqq 1$.

Leibniz' formula gives

$$\|P\nabla^{pm}u\|_{(p+1)\delta}\leqq\|\nabla^{pm}Pu\|_{(p+1)\delta}+\sum_{r=1}^{pm}\binom{pm}{r}\|P^{[r]}\nabla^{pm-r}u\|_{(p+1)\delta}$$