155. *Local Times on the Boundary for Multi-Dimensional Reflecting Diffusion*

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1. In connection with the study of the multi-dimensional diffusion processes with general boundary conditions, T. Ueno [8] introduced the notion of *the Markov process on the boundary concerning the diffusion.* He made a conjecture that this process would be obtained by a suitable change of time scale from the diffusion. As a typical case, for the reflecting Brownian motion on the half space he defined the local time on the boundary and showed that this is suitable as the time change function.

In this paper we shall define the local times on the boundary for the reflecting diffusion in multi-dimensions. This is the generalization of the local time at a single point for one-dimensional diffusion defined by P. Lévy [6], H. Trotter [7] and K. Itô-H. P. McKean, Jr. [3]. The results were published in [2] in mimeographed form with detailed proofs. Applications to the Markov processes on the boundary and diffusions with more general boundary conditions will appear in a paper by the first author.

We would like to thank N. Ikeda and T. Ueno. Our work owes much to them.

2. **The reflecting diffusion.** Let D be a domain with compact closure $\overline{D}$ in an N-dimensional orientable manifold of class C^∞. Assume that D has non-empty boundary ∂D consisting of a finite number of components each of which is an $(N-1)$-dimensional hypersurface of class C^3. We denote the local coordinate of the point x as $(x^1,\cdots,x^N)$. Let A be a second-order elliptic differential operator:

$$Au(x)=\frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\left(a^{ij}(x)\sqrt{a(x)}\frac{\partial u(x)}{\partial x^j}\right)+b^i(x)\frac{\partial u}{\partial x^i}, \tag{2.1}$$

where $a^{ij}(x)$ and $b^i(x)$ are contravariant tensors on $\overline{D}$ of class C^3, $a^{ij}(x)$ is symmetric and strictly positive definite for each $x\in\overline{D}$, and $a(x)=\det(a^{ij}(x))^{-1}$. In (2.1) we used the summation convention in differential geometry and $Au(x)$ is independent of the choice of local coordinates. The (inner) normal derivative is defined as

$$\frac{\partial u(x)}{\partial n}=\frac{1}{\sqrt{a^{NN}(x)}}a^{Ni}(x)\frac{\partial u(x)}{\partial x^i},\qquad x\in\partial D, \tag{2.2}$$

when in a neighborhood of x