38. A Theorem of Bari on the Completeness of Orthonormal Systems

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The purpose of the present note is to give an another proof of the following

THEOREM. *If $\{\varphi_n\}$ is a complete orthonormal system of a Hilbert space, and if $\{\psi_n\}$ is an another orthonormal system such as*

$$(1) \qquad \sum_{n=1}^{\infty}||\varphi_n-\psi_n||^2<\infty,$$

then $\{\psi_n\}$ is complete too.

The theorem is established by Nina Bari in 1941, according to her obituary note.[1] K. Iséki, in a note [2] published in these Proceedings, summarized several extensions of her theorem due to several authors. Recently, G. Birkhoff and G.-C. Rota [1] reproduced the theorem in a connection with the Sturm-Liouville expansions.

In Birkhoff-Rota's proof, the following lemma plays a central role:

LEMMA. *Under the hypothesis of the theorem, if m is a natural number such as*

$$(2) \qquad \sum_{n=m+1}^{\infty}||\varphi_n-\psi_n||^2<1,$$

then the sequence of vectors

$$(3) \qquad \varphi_1, \varphi_2, \cdots, \varphi_m, \psi_{m+1}, \psi_{m+2}, \cdots$$

is complete in the sense that no non-zero vector is orthogonal to (3).

Birkhoff-Rota's proof of the lemma is a simple application of the Parseval relation. In the present note, we shall give an alternative proof basing on the invertibility of an operator U defined by

$$(4) \qquad Ux=\sum_{n=1}^{m}\alpha_n\varphi_n+\sum_{n=m+1}^{\infty}\alpha_n\psi_n \quad \text{for} \quad x=\sum_{n=1}^{\infty}\alpha_n\varphi_n.$$

We can easily obtain that U is a bounded operator which satisfies

$$||I-U||^2\leqq||I-U||_2^2=\sum_{n=1}^{\infty}||(I-U)\varphi_n||^2=\sum_{n=m+1}^{\infty}||\varphi_n-\psi_n||^2<1,$$

since the uniform norm $||T||$ of an operator T is not greater than the Schmidt norm $||T||_2$ (e.g. [3]). Hence U has an inverse, so that U has the dense range which is spanned by (3). This shows that (3) is complete.

In the remainder of our proof, we shall employ a method inspired

1) Russian Mathematical Survey, **17**, 119–131 (1962). Unfortunately, Bari's original papers are unavailable to the present author.