142. On the Total Regularity of Riemann Summability

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§ 1. A method of summation is said to be regular if it assigns to every convergent series its actual value. If it furthermore assigns the value $+\infty$ to every series which diverges to $+\infty$, it is said to be totally regular. In this paper we shall consider the total regularity of Riemann summability. Throughout this paper, p denotes a positive integer. A series $\sum_{n=1}^{\infty} a_n$ is said to be summable (R, p) to s if the series in

$$f_p(t)=\sum_{n=1}^{\infty} a_n \left(\frac{\sin nt}{nt}\right)^p$$

converges in some interval $0<t<t_0$ and $f_p(t)\to s$ as $t\to 0+$. A series $\sum_{n=1}^{\infty} a_n$, with its partial sum s_n, is said to be summable (R_p) to s if the series in

$$F_p(t)=C_p^{-1}t\sum_{n=1}^{\infty} s_n \left(\frac{\sin nt}{nt}\right)^p,$$

where

$$C_p=\int_0^{\infty} u^{-p}\sin^p u\, du,$$

converges in some interval $0<t<t_0$ and $F_p(t)\to s$ as $t\to 0+$. It is well-known that the methods (R, p) and (R_p) are regular when $p\geqq 2$, while the methods $(R, 1)$ and (R_1) are not regular. (See, for example, [2]). But, concerning the total regularity of Riemman summability, the S.C. Lee's result [4] seems to be the only one. He proved that the method $(R, 2)$ is not totally regular.

§ 2. We shall first prove the following theorem.

THEOREM 1. *The method (R, p) is not totally regular when $p\geqq 2$. More precisely, given a monotone increasing sequence $\{W_n\}$ tending to $+\infty$ such that $W_n\, n^{-p}\to 0$ as $n\to\infty$, there exists a series $\sum_{n=1}^{\infty} a_n$ with $|a_n|\leqq 2W_n/n$ for all n, such that*

$$\sum_{n=1}^{\infty} a_n=+\infty \quad \text{and} \quad \liminf_{t\to 0+}\sum_{n=1}^{\infty} a_n\left(\frac{\sin nt}{nt}\right)^p=-\infty.$$

PROOF. We shall choose a sequence $\{N_k\}$ such that $N_1=1$, $2N_{k-1}<N_k$, and $N_k/25=$an integer when $k=2,3,4,\cdots$, and define a series $\sum_{n=1}^{\infty} a_n$ such that