203. *Decompositions of Generalized Algebras. I*

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In an unpublished paper [5],*) the author proposed an organic unification and generalization of the theories of G. Birkhoff's universal algebras [1], A. Tarski's relational systems [6], and G. Grätzer's multialgebras [3] (further, [2], [4]). Even under this very general setting, one is able to recapture the homomorphism theorems, the isomorphism theorems, and the Schreier-Jordan-Hölder theorems of algebra.

The unification was achieved by defining a *generalized algebra* (or simply a *genalgebra*) as a system $\mathfrak{S}=\langle G, o_1, \cdots, o_n, A\rangle$ consisting of a pair of sets G and A and a family (which may be finite or infinite) of (finitary or infinitary) functions

$$o_i : G^{m_i} \rightarrow A$$

$(i=1, \cdots, n)$ called operations. Thus, we have universal algebras when $A=G$; relational systems when $A=\{T, F\}$; multialgebras when $A=2^G$; and related universal algebras when $A=G\cup\{T, F\}$. The n-tuple $(m_1, \cdots, m_n)$ is called the *type* of the genalgebra. If $K\subseteq G$ and $C\subseteq A$ such that for each $i=1, \cdots, n$ and all elements $x_1, x_2, \cdots, x_{m_i}\in K$ we also have $o_i(x_1, x_2, \cdots, x_{m_i})\in C$, then $\mathcal{K}=\langle K, o_1, \cdots, o_n, C\rangle$ is said to be a *sub-genalgebra* of $\mathfrak{S}$. When C moreover is minimal, that is, when

$$C=\bigcup_{i=1}^{n} o_i(K, K, \cdots, K),$$

$\mathcal{K}$ is said to be a *reduced genalgebra.*

Given any other genalgebra $\mathcal{H}=\langle H, o'_1, \cdots, o'_n, B\rangle$ of the same type as $\mathfrak{S}=\langle G, o_1, \cdots, o_n, A\rangle$, a *homomorphism* from $\mathfrak{S}$ to $\mathcal{H}$ is a pair (h, k) of functions $h: G\rightarrow H$ and $k: A\rightarrow B$ such that for all $i=1, \cdots, n$, the following holds

$$k(o_i(x_1, x_2, \cdots, x_{m_i}))=o'_i(h(x_1), h(x_2), \cdots, h(x_{m_i}))$$

for all $x_1, x_2, \cdots, x_{m_i}\in G$. When both h and k are onto and one-to-one functions, then (h, k) is called an *isomorphism.* A *congruence* in the genalgebra $\mathfrak{S}$ is a pair (θ, φ) of equivalence relations θ on G and φ on A such that for each $i=1, \cdots, n$, if $(x_j, y_j)\in\theta$ for $j=1, 2, \cdots, m_i$, then also $(o_i(x_1, x_2, \cdots, x_{m_i}), o_i(y_1, y_2, \cdots, y_{m_i}))\in\varphi$. It should be noted that if (h, k) is a homomorphism of $\mathfrak{S}$ into $\mathcal{H}$, then (θ, φ) with $\theta=hh^{-1}$ and $\varphi=kk^{-1}$ is a congruence on $\mathfrak{S}$ (called the *kernel* of the homomorphism (h, k)). A congruence (θ, φ) on $\mathfrak{S}$ defines a new

*) For the references, see the list at the end of the following article.