71. On Holomorphic Markov Processes

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(Comm. by Zyoiti SUETUNA, M.J.A., April 12, 1966)

Under appropriate regularity conditions, a temporally homogeneous Markov process is associated with a contraction semi-group $\{T_t;\, t\geqq 0\}$ of class (C_0) [1] in a suitable Banach space X. In certain cases where X are complex Banach spaces, T_t admits a holomorphic extension T_λ given by strongly convergent Taylor series for all $x\in X$:

$$(1)\qquad T_\lambda x=\sum_{n=0}^{\infty}\frac{(\lambda-t)^n}{n!}T_t^{(n)}x \text{ for } \frac{|\lambda-t|}{t}\leqq \text{some positive constant } C,$$

the existence of the n-th strong derivative $T_t^{(n)}x$ in x of $T_t x$ being assumed for any $t>0$ and any $x\in X$ $(n=1, 2, \cdots)$. Such is the case of the semi-group

$$(2)\qquad \begin{aligned}(T_t f)(x)&=(2\pi t)^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{-|x-y|^2/2t}f(y)dy &&(t>0),\\ &=f(x) &&(t=0)\end{aligned}$$

in the Banach space $C[-\infty, \infty]$ of bounded uniformly continuous, complex valued functions $f(x)$ on $(-\infty, \infty)$ endowed with the maximum norm. Suggested by this example, we shall call a Markov process a *holomorphic Markov process* if the associated semi-group T_t admits a holomorphic extension T_t of the form given in (1). This notion seems to be of some interest. For instance, we can prove

Proposition. *Let a semi-group T_t with the infinitesimal generator A be associated with a holomorphic Markov process through*

$$(3)\qquad (T_t f)(x)=\int P(t, x, dy)f(y), \qquad f\in X$$

where $P(t, x, dy)$ is the transition probability of this process. Suppose that $T_{t_0}f_0=0$ for some $t_0>0$ and $f_0\in X$. Then $f_0=0$.

Proof. We have $A^n T_{t_0}f_0=T_{t_0}^{(n)}f_0=0$ $(n=0, 1, \cdots)$ by the linearity of A. Hence, by Taylor expansion (1), we see that $T_\lambda f_0=0$ for $|\lambda-t|/t\leqq C$. Repeating the argument, we easily see that $T_t f_0=0$ for all $t>0$ and so $f_0=s-\lim_{t\downarrow 0} T_t f_0=0$.

There are abundant examples of holomorphic Markov processes. In fact, the fractional power [2] $\hat{A}_\alpha$ $(0<\alpha<1)$ of the infinitesimal generator A of a contraction semi-group T_t of class (C_0) generates a constraction semi-group $\hat{T}_{t,\alpha}$ of class (C_0) which admits a holomor-