67. *Relations between Volumes and Measures*

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(Comm. by Kinjirô KUNUGI, M.J.A., April 12, 1967)

Introduction. A function v defined on a family V of sets of a space X is called a *volume* if the following two conditions are satisfied:

(1) The family V is a prering, that is the family is non-empty and if $A, B \in V$ then $A \cap B \in V$ and

$$A \backslash B = C_1 \cup \cdots \cup C_k,$$

where $C_j \in V$ are disjoint sets.

(2) The function v is non-negative, finite-valued, and countably additive on the prering V.

A volume v is called *upper complete* if the condition $A_n \in V$ and $\sum_n v(A_n) < \infty$ implies $A = \bigcup_n A_n \in V$. If in addition the condition $A \subset B \in V$ and $v(B) = 0$ implies $A \in V$, then the volume v is called *complete*.

In §1 we investigate upper complete volumes. The main result of the section is that upper complete volumes are in 1-1 correspondence with δ-finite measures. In this section we also establish the existence of *minimal extensions* of upper complete volumes to measures.

In §2 we prove that for every volume v there exists the *smallest complete measure* being an extension of the volume v. This result permits us to prove the classical theorem on extension of volumes. Namely if v is a volume on a prering V and M is the smallest σ-ring containing V, then there exists one and only one measure μ on M being an extension of the volume v. It is established that the completion of the measure μ_c yields the *smallest complete measure* being an extension of the volume v.

It is also established that for every volume v there exists the *smallest upper complete volume* being an extension of the volume v. The existence of the smallest complete volume satisfying this condition was established in [9].

§1. Relations between upper complete volumes and measures.

Theorem 1. *Let v be an upper complete volume on V and let M_0 be the family of all sets of the form $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in V$. Then the family M_0 is a sigma-ring.*

Theorem 2. *Let v be an upper complete volume on V and let*