132. On the Class of Paranormal Operators

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Introduction. In this paper we discuss a class of non-normal operators. We call a bounded linear operator T on a Hilbert space H *paranormal* if $\| T^2x\| \geqq \| Tx\|^2$ for every unit vector x in H. In [4] this is named an operator of class (N). It is easily known that this class includes hyponormal operators and is included in the class of normaloid operators.*) We show these inclusion relations are proper and hence paranormal operators constitute a new class broader than hyponormal operators and narrower than normaloid operators.

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1. **Lemma 1.** *Let T be a paranormal operator, then*

(1) $\| T^3x\| \geqq \| T^2x\| \cdot \| Tx\|$ *for every unit vector x in H.*

Proof. For a unit vector x in H, we may assume $Tx\neq 0$.

$$\| T^3x\| = \| Tx\| \cdot \left\| T^2\frac{Tx}{\| Tx\|}\right\| \geqq \| Tx\| \cdot \left\| T\frac{Tx}{\| Tx\|}\right\|^2$$

$$=\frac{\| T^2x\|^2}{\| Tx\|} \geqq \frac{\| T^2x\| \cdot \| Tx\|^2}{\| Tx\|} = \| T^2x\| \cdot \| Tx\| \qquad \text{q.e.d.}$$

Lemma 2. *Let T be a paranormal operator, then*

(P_k) $$\| T^{k+1}x\|^2 \geqq \| T^kx\|^2 \cdot \| T^2x\|$$

for a positive integer $k\geqq 1$ and every unit vector x in H.

Proof. For the case $k=1$

$$\| T^2x\|^2 = \| T^2x\| \cdot \| T^2x\| \geqq \| Tx\|^2 \cdot \| T^2x\|$$

and (P_1) is clear. Now suppose that (P_k) is valid for k and we assume $\| Tx\| \neq 0$, then

$$\| T^{k+2}x\|^2 = \| Tx\|^2 \left\| \frac{T^{k+1}Tx}{\| Tx\|}\right\|^2 \geqq \| Tx\|^2 \left\| T^k\frac{Tx}{\| Tx\|}\right\|^2 \left\| T^2\frac{Tx}{\| Tx\|}\right\|$$

$$= \| T^{k+1}x\|^2 \cdot \frac{\| T^3x\|}{\| Tx\|} \geqq \| T^{k+1}x\|^2 \cdot \| T^2x\|$$

by (1) of Lemma 1 and (P_k). So (P_{k+1}) is valid and the proof is complete by the mathematical induction. q.e.d.

Theorem 1. *If T is a paranormal operator, then T^n is paranormal for every integer $n\geqq 1$.*

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*) An operater T is said to be hyponormal if $T^*T \geqq TT^*$ and normaloid if $\| T^n\| = \| T\|^n$, (see definition 1).