155. On Some Integral Equations with Normal Integral Operators

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In the present paper we deal with the construction and the function theoretical properties of solutions of some integral equations with normal integral operators.

Definitions of notations. Let Δ be a Lebesgue ρ-measurable set of finite or infinite measure in m-dimensional real Euclidean space R_m; let $L_2(\Delta, \rho)$ be the Lebesgue functionspace; let $\{\varphi_\nu(x)\}_{\nu=1,2,3,\dots}$ and $\{\psi_\mu(x)\}_{\mu=1,2,3,\dots}$ be both incomplete orthonormal systems such that the union of them forms a complete orthonormal system in $L_2(\Delta, \rho)$; let $((\beta_{ij}))$ be the bounded normal operator in the Hilbert coordinate space l_2 corresponding to an infinite bounded normal matrix (β_{ij}) with $\sum_{j=1}^{\infty}|\beta_{ij}|^2 \neq |\beta_{ii}|^2 > 0$ $(i=1, 2, 3, \cdots)$; let $((\beta_{ij}^{(p)}))=((\beta_{ij}))^p$ $(p=1, 2, 3, \cdots, n)$ where $\beta_{ij}^{(1)}=\beta_{ij}$ $(i, j=1, 2, 3, \cdots)$; let $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ be any infinite bounded sequence of complex scalars; and let N_p be integral operators defined by

$$N_p h(x)=\sum_{\nu=1}^{\infty}\lambda_\nu^p\int_\Delta h(y)\overline{\varphi_\nu(y)}d\rho(y)\cdot\varphi_\nu(x)+c^p\sum_{\mu=1}^{\infty}\left\{\int_\Delta h(y)\overline{\psi_\mu(y)}d\rho(y)\cdot\sum_{j=1}^{\infty}\beta_{\mu j}^{(p)}\psi_j(x)\right\}$$
$$(p=1, 2, 3, \cdots, n;\ h(x)\in L_2(\Delta, \rho)),$$

where c is an arbitrarily given complex constant. Then, as we discussed before [1], N_1 is a bounded normal operator in $L_2(\Delta, \rho)$ and $N_p=N_1^p$.

Theorem 1. *Let $g(x)$ be any given function in $L_2(\Delta, \rho)$ such that it consists of all of $\varphi_\nu(x)$, $\psi_\mu(x)$; let $\zeta_p(p=1, 2, 3, \cdots, n)$ be the roots of the equation $\lambda^n+\sum_{p=1}^{n}a_p\lambda^{n-p}=0$ with complex coefficients a_p; let $\{\lambda_\nu\}$ be everywhere dense on an open or a closed rectifiable Jordan curve; let $\sup_\nu|\lambda_\nu|>|c|\left\{\sum_{i,j=1}^{\infty}|\beta_{ij}|^2\right\}^{\frac{1}{2}}<\infty$; and let $\sigma=\max_p\{|\zeta_p|\cdot\sup_\nu|\lambda_\nu|\}$. Then the integral equation*

$$(1)\qquad \lambda^n f(x)+\sum_{p=1}^{n}a_p\lambda^{n-p}N_p f(x)=g(x)\qquad (\sigma<|\lambda|<\infty)$$

has a uniquely determined solution

$$(2)\qquad f_\lambda(x)=\sum_{\nu=1}^{\infty}c_\nu\prod_{p=1}^{n}(\lambda-\zeta_p\lambda_\nu)^{-1}\varphi_\nu(x)+\frac{1}{\lambda^n}\left\{g(x)-\sum_{\nu=1}^{\infty}c_\nu\varphi_\nu(x)\right.$$
$$\left.+\sum_{k=1}^{\infty}\frac{1}{\lambda^k}\kappa_k(\zeta_1, \zeta_2, \cdots, \zeta_n)c^k\sum_{\mu=1}^{\infty}(g, \psi_\mu)\sum_{j=1}^{\infty}\beta_{\mu j}^{(k)}\psi_j(x)\right\}\in L_2(\Delta, \rho),$$