154. On Some Generalised Solution of a Nonlinear First Order Hyperbolic Partial Differential Equation

By Takaaki NISHIDA

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We consider the following Cauchy problem in $t\geq 0$, $-\infty<x<+\infty$.

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=h(u) \tag{1}$$

$$u(0, x)=u_0(x), \tag{2}$$

where $f(u)\in C^2$, $h(u)\in C^1$ and $u_0(x)\in L^\infty$.
First we assume that $f_{uu}(u)\geq\delta>0$ for $\forall u$.

Oleinik [1] proved the uniqueness and existence theorem of the generalised solution for Cauchy problem $u_t+f(t, x, u)_x=g(t, x, u)$ with (2) under the condition $f_{uu}\geq$ const. >0 and $|g_u(t, x, u)|\leq$ const. Here we consider the case that $|g_u(t, x, u)|\leq$ const. is not satisfied and see that the uniqueness and existence theorem is valid for some case under the following definition of the generalised solution.

We call $u(t, x)$ the generalised solution of (1)(2), which satisfies the following:

i) $u(t, x)$ is a measurable and locally bounded function.

ii) for arbitrary continuously differentiable function $\varphi(t, x)$ with compact support

$$\iint_{t\geq 0}\left[u\frac{\partial\varphi}{\partial t}+f(u)\frac{\partial\varphi}{\partial x}+h(u)\varphi\right]dt\,dx+\int_{-\infty}^{+\infty}\varphi(0, x)u_0(x)dx=0. \tag{3}$$

iii)

$$\frac{u(t, x_1)-u(t, x_2)}{x_1-x_2}<K(t, x_1, x_2), \tag{4}$$

where $K(t, x_1, x_2)$ is continuous in $t>0$, $-\infty<x_1, x_2<+\infty$.

§1. Uniqueness Theorem. We have the following uniqueness theorem.

Theorem. *The generalised solution $u(t, x)$ of (1)(2) is unique under the following estimate.*

$$\begin{aligned} &-\beta(t)\leq u(t, x)\leq\alpha(t, x) && \text{for}\quad t\geq 0,\quad x\geq 0,\\ &-\alpha(t, -x)\leq u(t, x)\leq\beta(t) && \text{for}\quad t\geq 0,\quad x\leq 0,\end{aligned} \tag{5}$$

where $\beta(t)$, $\alpha(t, x)$ are nonnegative and continuous in $\{t\geq 0\}$, $\{t\geq 0, x\geq 0\}$ respectively.

This can be proved by a slight modification of the argument in [1] th. 1. Following it, let us assume that there exist two generalised solutions $u_1(t, x)$ $u_2(t, x)$ satisfying (5). It is sufficient to see that for any $F(t, x)\in C^1$ such that there exist $T>\alpha>0$, $X>0$ (may depend