47. On a Kind of Uniqueness of Set-Entourage Uniformities for Function Spaces

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Let $\mathfrak{W}_p$, $\mathfrak{W}_c$, and $\mathfrak{W}_u$ be respectively the uniformity of pointwise, compact, and uniform convergence on a family of mappings of a uniform space into another uniform space. Usually uniformities $\mathfrak{W}_p$ and $\mathfrak{W}_u$ are simpler than $\mathfrak{W}_c$ for our use, but defective in some respect —$\mathfrak{W}_p$ is in many ways unnatural (cf. [3], p. 219) and $\mathfrak{W}_u$ can be applied only to families of uniformly continuous mappings when the continuity of operation of mapping composition is required (cf. Theorems 4 and 5 of [2]). On the other hand, as for $\mathfrak{W}_c$ there are no such defects as above and many desirable properties have been found in the literature. It seems to us that $\mathfrak{W}_c$ is the most natural set-entourage uniformity for families of mappings.

Let X be, for example, any locally euclidean, uniformly locally connected, metric space or any convex subset of a normed space, and let $\mathfrak{C}$ be the family of all continuous mappings of X into itself. The purpose of this note is to show that any set-entourage uniformity on $\mathfrak{C}$ must coincides with $\mathfrak{W}_c$, if the joint continuity and the continuity of operation of mapping composition are required under the uniform topology. This is an answer to the problem proposed in [2].

We begin with the definition of two terms used in the main theorem.

Definition 1. A uniform space X endowed with a uniformity $\mathfrak{U}$ is *uniformly deformable* if for any entourage $U \in \mathfrak{U}$ there exists an entourage $U^* \in \mathfrak{U}$ as follows: for any two U^*-close points p and q, there exists a continuous mapping f of X into itself such that $f(p)=q$ and $(x, f(x)) \in U$ for any $x \in X$.

Examples. The following spaces i), ii), $\cdots$, v) are uniformly deformable uniform spaces, whereas the space vi) is a manifold that is not uniformly deformable: i) locally euclidean, uniformly locally connected, uniform spaces, ii) locally euclidean, compact, uniform spaces, iii) convex subsets of a normed space, iv) the set of all rational points in a euclidean space, v) discrete uniform spaces, and vi) the set of all points (x, y) in the euclidean plane such that $(x^2+1)y^2>x$. (The uniformities of iii), iv), and vi) are the usual ones.)