99. Generalizations of the Alaoglu Theorem with Applications to Approximation Theory. II

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We will use the same notations as those in Part I.

7. **Theorem.** *Let $E_1, \cdots, E_n, E, F, X$, and Q have the same meaning as in Theorem* 3. Let

$$Y=\{[\lambda_1 x_1, \cdots, \lambda_n x_n] : |\lambda_i|\le 1, i=1, \cdots, n, [x_1, \cdots, x_n]\in X\}$$
$$Z_i=\{[y_1, \cdots, y_{i-1}, \lambda y_i+(1-\lambda)y_i', \cdots, y_n] : 0\le\lambda\le 1,$$
$$[y_1, \cdots, y_{i-1}, y_i, y_{i+1}, \cdots, y_n]\in Y$$
$$[y_1, \cdots, y_{i-1}, y_i, y_{i+1}, \cdots, y_n]\in Y\}$$
$$[X]=\cup\{Z_i : i=1, \cdots, n\}.$$

Suppose that 0 lies in the interior of the closure of $[X]$:

(C) $\qquad 0\in \operatorname{int} \overline{[X]}.$

Then, for each $k\ge 0$, the set $\{A\in b(E, F) : ||A-Q||_X\le k\}=S_k$ is σ-compact and σ-closed.

Proof. The proof is very similar to that of Theorem 3. Thus, for all $x\in X$ and all $A\in S_k$,

$$||Ax||\le ||Q||_X+k.$$

This inequality is valid if x ranges over the sets X, Y, Z_i $(i=1, \cdots, n)$ $[X]$ and, by continuity, $\overline{[X]}$. By Condition (C) the set contains an open sphere with radius $2r>0$. Then, for each $y\in E$,

$$||Ay||=||A\left(\frac{||y||}{r}\frac{r}{||y||}\right)||\le\frac{||y||^n}{r^n}(||Q||_X+k)\equiv k'||y||^n.$$

It follows that

$$S_k\subseteq \Pi_y\{f\in F : ||f||\le k'||y||^n\}$$

where the product on the right is compact in the product topology. By arguments similar to those in the proof of Theorem 3, we can easily show that any net in S_k has a subnet which converges in the σ-topology to an element of S_k, thus proving that S_k is σ-compact. Since a net in $b(E, F)$ converges to at most one limit in the σ-topology, the space is Hausdorff. Consequently, the σ-compactness implies the σ-closedness.

8. **Corollary.** *Let E_1 and E_2 be normed linear spaces. Let X_1 be a subset of E_1 such that* 0 *lies in the interior of the closed convex balanced extension of X_1. Let Q be a set-valued bounded map of X_1 into the dual space E_2^* of E_2. Then, Q has a best approximation in any τ-closed subset of $B(E_1, E_2^*)$.*