35. The Product of M-Spaces need not be an M-Space

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The notion of M-spaces has been introduced by K. Morita [1] and from that time on many interesting properties of M-spaces have been obtained by Morita and others. But a still unsolved problem is whether the cartesian product of M-spaces must be an M-space. The purpose of this note is to answer this question negatively.

We assume that all spaces are completely regular T_1-spaces and denote by βX and vX the Stone-Čech compactification and the Hewitt realcompactification of X respectively [2]. If X is not pseudocompact, then it is well known that $\beta X - vX \neq \varnothing$. A space X is said to be an *M-space* if there exists a normal sequence $\{\mathfrak{A}_n\,;\ n=1, 2, \cdots\}$ of open coverings of X satisfying the condition (M) below:

(M) If $\{K_n\}$ is a sequence of non-empty subsets of X such that $K_{n+1} \subset K_n$, $K_n \subset \mathrm{St}(x_0, \mathfrak{A}_n)$ for each n and for some fixed point x_0 of X, then $\cap \bar{K}_n \neq \varnothing$.

Theorem 1. *Suppose that X is not pseudocompact and P and Q are disjoint non-empty subsets of $\beta X - X$. If $X \cup P$ and $X \cup Q$ are countably compact, then $A \times B$ is not an M-space where $A = X \cup P \cup \{x^*\}$, $B = X \cup Q \cup \{x^*\}$ and x^* is an arbitrary point contained in $\beta X - vX$.*

Proof. Since A and B are countably compact these spaces are M-spaces. x^* belongings to $\beta X - vX$, there exists a continuous function f on βX such that $f > 0$ on X and $f(x^*) = 0$. It is obvious that

$$\cup (X \cap Z_n) = X \quad \text{where} \quad Z_n = \{x\,;\ f(x) \geqq 1/n,\ x \in \beta X\}.$$

Now suppose that $A \times B$ is an M-space. Then there exists a normal sequence $\{\mathfrak{A}_n\,;\ n=1, 2, \cdots\}$ of open coverings of $A \times B$ satisfying the condition (M). Let us put $s^* = (x^*, x^*)$. Since $\mathrm{St}(s^*, \mathfrak{A}_n)$ is an open set of $A \times B (\subset \beta X \times \beta X)$, there is an open set U_n (in βX) containing x^* such that

$$U_n \cap Z_n = \varnothing, \qquad \mathrm{cl}_{\beta X} U_{n+1} \subset U_n$$
$$\text{and} \quad (A \times B) \cap (U_n \times U_n) \subset \mathrm{St}(s^*, \mathfrak{A}_n).$$

As is well known every point of $\beta X - X$ is not G_δ in βX and hence $\cap U_n$ contains a point $y^* (\neq x^*)$ of $\beta X - X$ (notice that $\cup Z_n \supset X$ and $U_n \cap Z_n = \varnothing$). $x^* \neq y^*$ leads to the existence of an open set V of βX containing y^* whose closure does not contain x^*. We denote by $\Delta(X)$ the diagonal set of $X \times X$ and by K_n the following set

$$(V \times V) \cap (U_n \times U_n) \cap \Delta(X).$$