# 155. *Representation of Certain Banach $*$-algebras*[*)]

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Let $A$ be a Banach $*$-algebra satisfying the condition: there exists a positive constant $\alpha$ such that

$$\alpha\|x^*\|\|x\|\leq\|x^*x\|$$

for every $x$ in $A$. The problem to realize such a Banach $*$-algebra as a $C^*$-algebra has been left to be solved after I. Kaplansky [3] asked whether or not every $C$-symmetric Banach $*$-algebra is symmetric. In the case when $A$ is commutative, R. Arens [1] had proven that it is a $B^*$-algebra under an equivalent norm, and then B. Yood [8] gave a partial answer to this problem by showing that a Banach $*$-algebra with the above condition is a $B^*$-algebra under an equivalent norm if $\alpha>c$ ($c$; the unique real root of the equation $4t^3-2t^2+t-1=0$).

The purpose of this note is to inform that this problem has been solved in the affirmative, and is to give a brief account of the proof. Our result is the following.

**Theorem.** *Let $A$ be a Banach $*$-algebra whose norm satisfies the condition $\alpha\|x^*\|\|x\|\leq\|x^*x\|$. Then it is homeomorphic and $*$-isomorphic to a $C^*$-algebra.*

By a $B^*$-algebra, we shall mean a Banach $*$-algebra with the condition $\|x^*x\|=\|x\|^2$. At the present time, it is well known that a $B^*$-algebra is isometrically $*$-isomorphic to a $C^*$-algebra, a uniformly closed $*$-algebra of operators on Hilbert space.

Throughout this paper we shall consider a (complex) Banach $*$-algebra with unit $e$ (the case without unit will be mentioned at the final step). Here we present a concise proof of the theorem which proceeds by stages. In the course of the representation of $B^*$-algebras (see the theorem of Fukamiya and Kaplansky [7; Theorem 4.8. 11], T. Ono [6] and J. Glimm-R. V. Kadison [2]), the problem one discussed for a long time was to extend the local $C^*$-property to the global one. Concerning our problem we are in the same situation as the case of $B^*$-algebras because Arens [1] tells us that our Banach $*$-algebras provide the local $C^*$-property. To clarify the essence of the proof we introduce a class of Banach $*$-algebras as follows. A Banach $*$-algebra $A$ is said to be *locally equivalent to a $C^*$-algebra* if every maximal

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