32. *L^p-theory of Pseudo-differential Operators*

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(Comm. by Kinjirô KUNUGI, M. J. A., Feb. 12, 1970)

Introduction. The L^2-theory of pseudo-differential operators has been studied in many papers, but we know very few papers which are concerned with L^p-theory. We say $g(x, \xi) \in S^m_{\rho,\delta}$, $0<\rho\leqq 1$, $0\leqq\delta$, when $g(x, \xi) \in C^\infty(R^n_x \times R^n_\xi)$ and for any α, β, there exists a constant $C_{\alpha,\beta}$ such that

$$|\partial^\alpha_x \partial^\beta_\xi g(x, \xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|}$$

where $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\beta=(\beta_1, \cdots, \beta_n)$ are multi-indices whose elements are non-negative integers, $\langle\xi\rangle=(1+|\xi|^2)^{\frac{1}{2}}$, and $\partial_{x_j}=\partial/\partial x_j$, $\partial_{\xi_j}=\partial/\partial\xi_j$, $j=1, \cdots, n$,

$$\partial^\alpha_x=\partial^{\alpha_1}_{x_1}\cdots\partial^{\alpha_n}_{x_n},\ \partial^\beta_\xi=\partial^{\beta_1}_{\xi_1}\cdots\partial^{\beta_n}_{\xi_n},\ |\alpha|=\alpha_1+\cdots+\alpha_n,$$

$|\beta|=\beta_1+\cdots+\beta_n$. For a pseudo-differential operator defined by the symbol of class $S^m_{\rho,\delta}$, the L^2-boundedness of the form $\|g(X, D_x)u\|_s \leqq C\|u\|_{m+s}$ was proved by Hörmander [2] and Kumano-go [4] in the case $0\leqq\delta<\rho\leqq 1$.

In the present paper we shall study the general L^p-theory for pseudo-differential operators of class $S^m_{1,\delta}$ in the case: $0\leqq\delta<1$ and $1<p<\infty$. Recently for operators of class $S^0_{1,\delta}$, Kagan [3] proved the L^p-boundedness: $\|p(X, D_x)u\|_{L^p} \leqq C\|u\|_{L^p}$ for $1<p\leqq 2$. Applying the theory in Kumano-go [5], we first prove the inequality $\|g(X, D_x)u\|_{p,s} \leqq C\|u\|_{p,m+s}$ for any real s and $1<p<\infty$ (which solves a problem of Hörmander in [2], p. 163, for the typical case $\rho=1$), and prove the theorems: the generalized Poincaré inequality, the invariance of the space $H_{p,s}$ under coordinate transformation and the a priori estimate for elliptic operators.

1. Definitions and fundamental lemmas.

We shall use the following notations:

$$\mathcal{S}=\{u(x) \in C^\infty(R^n)\,;\ \lim_{|x|\to\infty} |x|^m|\partial^\alpha_x u(x)|=0 \text{ for any } m \text{ and } \alpha\}.$$

$\mathcal{S}'$ denotes the dual space of $\mathcal{S}$. For $u \in \mathcal{S}$, we define the Fourier transform of u by $\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx$, $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$. For any real s we define an operator $\langle D_x\rangle^s$: $\mathcal{S}\to\mathcal{S}$ by

$$\langle D_x\rangle^s u(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}\langle\xi\rangle^s\hat{u}(\xi)d\xi.$$

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