58. Boundary Value Problems for Some Degenerate Elliptic Equations of Second Order with Dirichlet Condition

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1. Introduction. Let Ω be a domain in R^n whose boundary is a smooth and compact hypersurface. We deal with the following differential operator defined in Ω:

$$A_\rho(x, D) = -\rho(r) \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial}{\partial x_k}\right) + \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j} + c(x) \tag{1.1}$$

where r denotes the distance from $x \in \bar{\Omega}$ to Γ, the boundary of Ω, and we assume that

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geqq \delta|\xi|^2 \qquad \text{for any real } n\text{-vector } \xi \quad (a_{jk} = \bar{a}_{kj}), \tag{1.2}$$

and $\rho(t)$ $(t \in \bar{R}^1_+)$ satisfies

1) $\rho(t) \in C^0(\bar{R}^1_+) \cap C^2(R^1_+)$ and $0 \leqslant \rho(t)$ with $\rho(t) = 0$ only at $t = 0$
2) $\rho(t)^{-1}$ is integrable in $(0, s)$ for any $s \geqq 0$, and $\rho''(t) \leqq 0$ near $t = 0$
3) $|\rho'(t)| \leqq C_1 t^{\alpha-1}$ and $|\rho''(t)| \leqq C_2 t^{\alpha-2}(0 < \alpha < 1)$ near $t = 0$
4) $\int_0^a t^{2\alpha-2}\int_0^t \rho(s)^{-1}dsdt$ and $\int_0^a \rho'(t)\rho(t)^{-1}\int_0^t \rho(s)^{-1}dsdt$ are finite

for any $a > 0$ and if Ω is unbounded, we assume moreover

5) when $t \to \infty$, $0 < K \leqq \rho(t)$ and $\rho'(t)$, $\rho''(t)$ remain bounded.

If we take a function to be equal to t^α near $t = 0$ as $\rho(t)$, we can see easily that it satisfies the above conditions.

For the coefficients of $A_\rho(x, D)$, we assume that $a_{jk}(x)$ and $b_j(x)$ are all in $\mathscr{B}^1(\bar{\Omega})$, and $c(x)$ in $C^0(\Omega)$ with $|c(x)| \leqq M|\rho'(r)|\rho(r)^{-1}$ near Γ, and if Ω is unbounded, we assume that $c(x)$ remains bounded as $|x| \to \infty$.

Now let us introduce some Hilbert spaces in which we develop our arguments.

Definition 1.1. We say $u(x)$ belongs to $L^2(\Omega, \rho^{-1})$ if and only if

$$\|u\|^2_{0,\rho^{-1}} = \int_\Omega |u(x)|^2 \rho(r)^{-1} dx \tag{1.3}$$

is finite.

Definition 1.2. $u(x)$ is said to be in $H^m(\Omega, \rho)$, if and only if

$$\|u\|^2_{m,\rho} = \int_\Omega \Big(\rho(r) \sum_{|\alpha|=2} |D^\alpha u|^2 + |u|^2\Big) dx \tag{1.4}$$

is finite.

One of our main results is

Theorem 1.1. *Under the conditions stated above, the equation*

$$\begin{cases} A_\rho(x, D)u + \lambda u = f(x) \\ u|_\Gamma = 0 \end{cases} \tag{1.5}$$