57. *A Construction for Idempotent Binary Relations*

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The problem of characterizing the idempotent elements of the semigroup $\mathfrak{R}_A$ of all binary relations over a set A is of interest, since the semigroup $\mathfrak{R}_A$ was a subject of numerous studies. This problem has been mentioned in [1]. Here we present a solution to this problem.

Let $\rho \in \mathfrak{R}_A$. The set of all $a \in A$ such that $(a, a_1) \in \rho$ or $(a_1, a) \in \rho$ for some $a_1 \in A$ is called the *field* of ρ and is denoted as $pr\rho$. ρ may be considered as a binary relation over its field.

If $\mathfrak{a} \subset A$, then $\Delta_\mathfrak{a}$ denotes the binary relation over A defined as follows: $(a_1, a_2) \in \Delta_\mathfrak{a}$ iff $a_1 = a_2$ and $a_1 \in \mathfrak{a}$.

A reflexive and transitive binary relation is called a *quasi-order* relation. An antisymmetric quasi-order relation is called an *order* relation.

Let ρ be a binary relation over a set I and $(A_i)_{i\in I}$ be a family of pairwise disjoint nonempty sets. Then the binary relation $\bigcup_{(i,j)\in\rho} (A_i \times A_j)$ over the set $\bigcup_{i\in I} A_i$ is called an *inflation* of ρ. It is known that *inflations of order relations are quasi-order relations and every quasi-order relation is a uniquely determined inflation of a uniquely determined (up to isomorphism) order relation.* Thus, the structure of quasi-order relations may be considered as known modulo order relations.

Let ρ be a quasi-order relation over a set A. Then $\varepsilon_\rho = \rho \cap \rho^{-1}$ is the *symmetric part* of ρ (here ρ^{-1} is the *converse* of ρ). Clearly, ε_ρ is an equivalence relation over A. An element $a \in A$ is called *ρ-strict* if the ε_ρ-class containing a is a singleton, i.e., if $(a, a_0), (a_0, a) \in \rho$ imply $a = a_0$. An element $a_1 \in A$ *covers* an element $a_2 \in A$ if $(a_1, a), (a, a_2) \in \rho$ imply $a = a_1$ or $a = a_2$ for every $a \in A$, and $(a_1, a_2) \in \rho$. Two elements a_1 and a_2 are called *ρ-neighbors* if a_1 covers a_2 or a_2 covers a_1. A subset $\mathfrak{a} \subset A$ is called *ρ-permissible*, if all elements of $\mathfrak{a}$ are ρ-strict and $\mathfrak{a}$ does not contain ρ-neighbors.

A binary relation σ is called a *pseudo-order* relation if $\sigma = \rho \backslash \Delta_\mathfrak{a}$ where ρ is a quasi-order relation and $\mathfrak{a}$ is a ρ-permissible subset. Here $\backslash$ is the set-theoretical difference. In this case ρ is called the *completion* of σ, and $\mathfrak{a}$ is called the *defect* of σ. Since $\rho = \sigma \cup \Delta_A$ and $\mathfrak{a} = A \backslash pr(\sigma \cap \Delta_A)$, the completion and defect of σ are uniquely determined. Notice that $pr\sigma$ need not be equal to A.

Theorem. *A binary relation is idempotent if and only if it is a*