113. *An Application of Serre-Grothendieck Duality Theorem to Local Cohomology*

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The purpose of this note is to prove the following theorem using the Serre-Grothendieck duality theorem and to derive two formulae from it. These formulae show that some algebro-geometric notions can be expressed with local cohomology.

Theorem. *Let X and S be locally noetherian preschemes of finite Krull dimension. Let $f: X \to S$ be a proper, smooth morphism of relative dimension n, and let $s: S \to X$ be a f-section. We identify the image of s with S and denote by $\mathcal{J}$ the sheaf of ideals in $\mathcal{O}_X$ defining the closed subprescheme S. Denote by $\omega_{X/S}$ the sheaf of n-th differential forms on X relative to f. Then for every coherent sheaf $\mathcal{F}$ on X and for every integer $p \geq 0$, there exists a functorial isomorphism*

$$(1) \qquad f_*(\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}^n_S(\omega_{X/S}))) \simeq \varinjlim_{k\geq 0} \mathcal{E}xt^p_{\mathcal{O}_S}(f_*(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} (\mathcal{O}_X/\mathcal{J}^{k+1})), \mathcal{O}_S).$$

$$\begin{tikzcd} X \arrow[d, "f"'] & S \arrow[l, "s"'] \arrow[dl, "id"] \\ S & \end{tikzcd}$$

Remark 1. We can eliminate the regularity condition for f using derived functors. We can treat even more general cases ([N]). The proof becomes, however, so complicated in spite of little merit of generalization.

Proposition (SGA II, exposé VI, Theorem 2.3). *Let X be a locally noetherian prescheme and let Y be a closed subprescheme of X defined by a coherent sheaf of ideals $\mathcal{J}$. Then for every coherent sheaf $\mathcal{F}$ on X and for every quasi-coherent sheaf $\mathcal{G}$ on X, there is a spectral sequence*

$$(2) \qquad \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}^q_Y(\mathcal{G})) \Rightarrow \varinjlim_{k\geq 0} \mathcal{E}xt^{p+q}(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} (\mathcal{O}_X/\mathcal{J}^{k+1}), \mathcal{G}).$$

Lemma. *Under the same hypothesis of the theorem, let $\mathcal{G}$ be a sheaf of abelian groups on X with support in S. Then for all $p > 0$,*

$$R^p f_*(\mathcal{G}) = 0.$$

Proof. On the category of sheaves of abelian groups whose supports are in S, it is obvious that the direct image functor f_* is the same as the restriction functor s^* under the above hypothesis. Since s^* is an exact functor and the canonical flasque resolution of $\mathcal{G}$ can be