110. $\bar{l}$-Spaces over Locally Convex Spaces*)

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1. In the previous note [3], we defined the l^p-space over a Banach space and used it for a study of polynomial maps of Banach spaces. It seems to be more useful to define a similar space for a locally convex topological vector space. In this note we shall do this.

Let E be a locally convex (topological vector) space and S be its (irreducible) spectrum of seminorms [2]. Then the n^{th} tensor power $E^{\otimes n}$ of E with the projective topology admits as its spectrum the irreducible hull of the set of seminorms $\{\mathfrak{p}^{\otimes n} \mid \mathfrak{p} \in S\}$ where $\mathfrak{p}^{\otimes n}$ is a seminorm defined by $\mathfrak{p}^{\otimes n}(x) = \inf\{\sum \mathfrak{p}(x_1^{(i)}) \cdots \mathfrak{p}(x_n^{(i)}) \mid x = \sum x_1^{(i)} \otimes \cdots \otimes x_n^{(i)}\}$ for $x \in E^{\otimes n}$. For any $p, 1 \leq p < \infty$, and for any $\mathfrak{p} \in S$, we define a real valued function $l^p\mathfrak{p}$ on the (algebraic) vector space $\bigoplus_{n=1}^{\infty} E^{\otimes n}$ by $l^p\mathfrak{p}(x) = (\sum \mathfrak{p}^{\otimes n}(x_n)^p)^{1/p}$ for $x = \sum x_n, x_n \in E^{\otimes n}$. It is clearly a seminorm. Let l^pS be the irreducible hull of seminorms $\{l^p\mathfrak{p} \mid \mathfrak{p} \in S\}$, we define a locally convex space $\bar{l}^pE$ to be the set $\{x = \sum x_n \mid x_n \in E^{\otimes n}$ and $l^p\mathfrak{p}(x) < \infty$ for any $\mathfrak{p} \in S\}$ with the spectrum l^pS, and $\bar{l}_s^pE$ to be its subspace of symmetric elements. Then the following properties are easily verified.

Proposition 1. *If E is a Frechet space, so are $\bar{l}^pE$ and $\bar{l}_s^pE$. If E is Frechet and nuclear, then we have $(\bar{l}^pE)' \cong \bar{l}^qE'$ and $(\bar{l}_s^pE)' \cong \bar{l}_s^qE'$ where E' is the strong dual of E and $1/p + 1/q = 1$.*

As usual, we have $\bar{l}^pE \subset \bar{l}^qE$ if $p \leqq q$, moreover we have

Theorem 1. *For any $p, q \geq 1$, $\bar{l}^pE \subset \bar{l}^qE$ and the inclusion is continuous.*

Lemma. *For any sequence $\{a_n\}$ of positive numbers with* $\lim a_n^{1/n} = 0$ *and for any real $s \geqq l$, we have $(\sum a_n)^s \leqq \sum (2^n a_n)^s$.*

This Lemma is easily verified.

Proof of Theorem 1. For any $\mathfrak{p} \in S_E$ and $x_n \in E^{\otimes n}$, we have $t\mathfrak{p} \in S_E$ for any $t > 0$ and $(t\mathfrak{p})^{\otimes n}(x_n) = t^n \mathfrak{p}^{\otimes n}(x_n)$, hence $x = \sum x_n \in \bar{l}^pE$ for some p if and only if $\lim (\mathfrak{p}^{\otimes n}(x_n))^{1/n} = 0$. Then $x \in \bar{l}^qE$ for any q. This means that $\bar{l}^pE$ and $\bar{l}^qE$ coincide with each other as sets. Let $p \geqq q \geqq 1$. Let $a_n = (\mathfrak{p}^{\otimes n}(x_n))^q$ for an element $x = \sum x_n \in \bar{l}^pE$ and a seminorm $\mathfrak{p} \in S_E$, then $s = p/q \geqq 1$, $\lim a_n^{1/n} = 0$ and $\mathfrak{p}^{\otimes n}(x_n)^p = a_n^s$ hence, by the above Lemma, we have $(l^q\mathfrak{p}(x_n))^p = (\sum \mathfrak{p}^{\otimes n}(x_n)^q)^{p/q} = (\sum a_n)^s \leqq \sum (2^n a_n)^s = \sum ((2\mathfrak{p})^{\otimes n}(x_n))^p = (l^p(2\mathfrak{p})(x))^p$. Let $\mathfrak{q}$ be any seminorm in l^qS_E, then

*) Dedicated to Professor A. Komatu on his sixtieth birthday.