215. On an Algebraic Model for von Neumann Algebras

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1. Recently N. Dinculeanu and C. Foiaş [2] introduced the concept of algebraic models for probability measures in their researches on conjugacy of measure preserving transformations.

Since the theory of von Neumann algebras is recognized as a non-commutative extension of the measure theory, we can expect that the theory of Dinculeanu and Foiaş has an analogue for von Neumann algebras. In the present note, we shall engage in this direction.

2. Let (Γ, φ) be a pair of a group Γ and a complex function φ of positive type defined on Γ. Then we shall call (Γ, φ) a *measure system* provided that $\varphi(\gamma)=1$ if and only if $\gamma=1$. Especially, in case that Γ is abelian, our notion coincides with that of Dinculeanu-Foiaş. Two measure systems (Γ, φ) and (Γ', φ') are said to be *isomorphic* if there exists an isomorphism ϕ of Γ onto Γ' such that $\varphi(\gamma)=\varphi'(\phi\gamma)$ for γ in Γ.

Now we shall introduce the notion of an algebraic model for a von Neumann algebra which is a modification of that of Dinculeanu-Foiaş:

Definition 1. Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ with a generating vector x. A measure system (Γ, φ) is an *algebraic model* for $\mathcal{A}$, if there exists an isomorphism J of Γ into the unitary group of $\mathcal{A}$ such that

(i) $J\Gamma$ generates $\mathcal{A}$,

and

(ii) $\varphi(\gamma)=(J\gamma x \mid x)$, for γ in Γ.

It is clear that the unitary group $\Gamma(\mathcal{A})$ itself is an algebraic model for $\mathcal{A}$ if x is separating.

Let us suppose that (Γ, φ) is a measure system. Since φ is positive definite, the theorem of Gelfand and Raikov (cf. [3; p. 393]) gives a unitary representation π of Γ on a Hilbert space $\mathfrak{H}_\varphi$ induced by φ such that

$$\varphi(\gamma)=(\pi(\gamma)\xi \mid \xi) \tag{1}$$

for every γ in Γ, where ξ is a generating vector for $\pi(\Gamma)$. Since $\varphi(\gamma)=1$ if and only if $\gamma=1$, π is an injective map. In fact, if $\pi(\gamma_1)=\pi(\gamma_2)$ for $\gamma_1, \gamma_2 \in \Gamma$, then $\pi(\gamma_1\gamma_2^{-1})=\mathrm{I}$, so that $\varphi(\gamma_1\gamma_2^{-1})=1$ by (1) or $\gamma_1=\gamma_2$. Let $\mathcal{A}(\Gamma, \varphi)$ denote the von Neumann algebra generated by $\{\pi(\gamma) \mid \gamma \in \Gamma\}$. Then we have the following theorem:

Theorem 1. *Let $\mathcal{A}$ be a von Neumann algebra acting on $\mathfrak{H}$ with*