17. *Operators Satisfying the Growth Condition* (G_1)

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1. This note is motivated by the following theorem by I. H. Sheth.

Theorem 1 [6]. *Let* $T=UR$, $R=(T^*T)^{1/2}$ *be an invertible hyponormal operator such that* U *is cramped, then* $0 \notin \overline{W(T)}$.

The purpose of this note is to prove a generalization of Theorem 1 to the case of operators satisfying the growth condition (G_1). The technique of [6] actually proves the following theorem.

Theorem 2. *Let* $T=UR$, $R=(T^*T)^{1/2}$ *be an invertible operator such that* T *satisfies* (G_1) *and* U *is cramped, then* $0 \notin \overline{W(T)}$.

In the case of normal operator, this was proved by Berberian [1]. Durszt [2] constructed an invertible operator T such that the unitary operator $U=T(T^*T)^{-1}$ is cramped and $0 \in \overline{W(T)}$.

2. In the following, an operator means a bounded linear operator on a Hilbert space. Let T be an operator on H, $\sigma(T)$ and $\sigma_a(T)$ denote the spectrum and the approximate point spectrum of T respectively. Let conv $\sigma(T)$ be the (automatically closed) convex hull of $\sigma(T)$. The numerical range $W(T)$ is the set $W(T)=\{(Tx,x): x\in H, \|x\|=1\}$. We write $\overline{W(T)}$ the closure of $W(T)$. T satisfies the condition (G_1) if

$$(G_1) \qquad \|(T-\alpha I)^{-1}\| \leq 1/d(\alpha, \sigma(T))$$

for all $\alpha \notin \sigma(T)$, where $d(\alpha,\sigma(T))$ is the distance from α to $\sigma(T)$. A unitary operator U is cramped if $\sigma(U)\subset\{e^{i\theta}: \theta_0<\theta<\theta_0+\pi\}$.

If T is hyponormal, T satisfies Condition (G_1). In fact, in this case $(T-\alpha I)^{-1}(\alpha \notin \sigma(T))$ is also hyponormal, hence

$$\|(T-\alpha I)^{-1}\|=1/\inf\{|\lambda-\alpha|: \lambda \varepsilon \sigma(T)\}=1/d(\alpha,\sigma(T)).$$

Let X be a compact convex set of the complex plane. A point $\lambda \in X$ is bare if there is a circle through λ such that no points of X lie outside this circle.

3. To prove Theorem 2, we use the following facts which are stated as lemmas.

Lemma 1. *If* U *is unitary,* U *is cramped if and only if* $0 \notin \overline{W(U)}$.

Proof. See [1: Lemma 3].

Lemma 2. *Let* T *be an operator which satisfies Condition* (G_1), *then every bare point* λ *of* $\overline{W(T)}$ *is contained in* $\sigma_a(T)$ *and has the following property:* $Tx_n-\lambda x_n\to 0$ $(n\to\infty)$ *if and only if* $T^*x_n-\bar{\lambda}x_n\to 0$ $(n\to\infty)$ *for a sequence* $\{x_n\}$ *of unit vectors.*