3. *A Note on Artinian Subrings*

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Throughout, A will represent a ring with the identity element 1, $J(A)$ the radical of A, and a subring of A will mean one containing 1. If S is a subset of A, $V_A(S)$ means the centralizer of S in A. A left A-module M is always unital and denoted by ${}_AM$.

The purpose of this note is to prove the following:

Theorem 1. *Let B be a subring of A such that ${}_BA$ is f.g. (finitely generated), and T a left Artinian subring of A containing B. Let $\bar{T}=T/J(T)$, and $\bar{B}=B+J(T)/J(T)$. If $\bar{T}=\bar{B}\cdot V_{\bar{T}}(\bar{B})$ and the left $\bar{T}$-module $\bar{A}=A/J(T)A$ is faithful then B is left Artinian.*

Our theorem contains evidently D. Eisenbud [3; Theorem 1 b)] and draws out J.-E. Björk [2; Theorem 3.4] as an easy corollary.

Lemma 1. *Let $M=Au_1+Au_2+\cdots+Au_n$ be a unital A-A-module such that $Au_i=u_iA$ and u_1 is left A-free. If for every non-zero ideal $\mathfrak{a}$ of A there holds $\mathfrak{a}M=M$, then A is two-sided simple.*

Proof. Without loss of generality, we may assume that $M\neq Au_1+\cdots+Au_{i-1}+Au_{i+1}+\cdots+Au_n$ for each $1<i\leqslant n$. We shall prove then by induction $M=Au_1\oplus\cdots\oplus Au_n$, which implies at once that A is two-sided simple. We set $M_k=Au_1+\cdots+Au_k$ for $1\leqslant k\leqslant n$. Evidently, $\mathfrak{a}_n=\{a\in A\mid au_n\in M_{n-1}\}$ is an ideal of A. If $\mathfrak{a}_n$ is non-zero then $M=\mathfrak{a}_nM=M_{n-1}$. This contradiction proves $M=M_{n-1}\oplus Au_n$. Next, assume that $M=M_k\oplus Au_{k+1}\oplus\cdots\oplus Au_n$ has been proved. It will be easy to see that $M_k\neq Au_1+\cdots+Au_{i-1}+Au_{i+1}+\cdots+Au_k$ for each $1<i\leqslant k$. If $\mathfrak{a}$ is a non-zero ideal of A then $\mathfrak{a}M_k\oplus\mathfrak{a}u_{k+1}\oplus\cdots\oplus\mathfrak{a}u_n=M_k\oplus Au_{k+1}\oplus\cdots\oplus Au_n$ implies at once $\mathfrak{a}M_k=M_k$. Hence, by the first step, we obtain $M_k=M_{k-1}\oplus Au_k$, which completes the induction.

Proposition 1. *Let $A=A_1\oplus\cdots\oplus A_n$, where A_i is a two-sided simple [Artinian simple] ring with the identity element e_i. Let B be a subring of A such that ${}_BA$ is f.g. If $A=B\cdot V_A(B)$ then $V_A(B)=V_1\oplus\cdots\oplus V_n$ and $B=B_1\oplus\cdots\oplus B_k$ ($k\leqslant n$), where V_i is Artinian simple and B_i is two-sided simple [Artinian simple].*

Proof. At first, we shall prove the case $A=A_1$. Evidently, $A=Bv_1+\cdots+Bv_s$ with $v_1=1$ and $v_2,\cdots,v_s\in V_A(B)$. As we can easily

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