38. *Properties of Ergodic Affine Transformations of Locally Compact Groups. III*

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Let G be an abelian group. An affine transformation S of G is a transformation of G onto itself of the form $S(x)=a+T(x)$, where $a \in G$ and T is an automorphism of G. In case G is a locally compact non-discrete topological group, it has been proved (cf. [1], [2], [3] and [4]) that if there exists a continuous affine transformation S of G which has a dense orbit then G is compact. In the present paper we shall study the structure of a discrete abelian group G which is covered by an orbit under an affine transformation S.

1. Theorems.

From now on, for simplicity, we say that an affine transformation S of G satisfies property $\mathcal{A}$ if $\{S^n(w)\,;\, n=0, \pm 1, \pm 2, \cdots\}=G$ for some $w \in G$.

Theorem 1. *Let G be an infinite abelian group. If G has an affine transformation $S(x)=a+T(x)$ satisfying property $\mathcal{A}$ then G is isomorphic with the additive group Z of the integers, a is a generator, and T is the identity transformation.*

Theorem 2. *Let G be a finite abelian group with order r. If 4 does not divide r, and G has an affine transformation $S(x)=a+T(x)$ satisfying property $\mathcal{A}$ then G is isomorphic with the cyclic group $Z(r)$ of order r, and a is a generator.*

2. Proof of Theorem 1.

Lemma 1. *If G has an affine transformation $S(x)=a+T(x)$ satisfying property $\mathcal{A}$ then G is finitely generated.*

Proof. Since $\{S^n(0)\,;\, n=0, \pm 1, \pm 2, \cdots\}=\{S^n(w)\,;\, n=0, \pm 1, \pm 2, \cdots\}=G$, $T(a)=S^k(0)$ for some integer k. If $k=0$ (resp. 1, or 2) then it is easy to check that $G=\{0\}$ (resp. $G=\{na\,;\, n=0, \pm 1, \pm 2, \cdots\}$, or $G=\{0\}$). If $k \geqq 3$, we see that $T^k(a)$ is in the subgroup H generated by $\{a, T(a), \cdots, T^{k-1}(a)\}$. It follows at once that

$$a \in T(H) \subset H,$$

and hence $T(H)=H$, and $S(H)=H$. This clearly assures that $G=H$, the required conclusion. A similar argument also applies in the case $k<0$, and so G is finitely generated.

Lemma 2. *If the additive group $Z^p (p \geqq 1)$ has an affine transfor-*