55. *A Note on Approximate Dimension*

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Mityagin has characterized nuclear spaces by the approximate dimension. In an F-space E, namely, E is nuclear iff the approximate dimension of E is zero. (It is known that the approximate dimension is zero if it is finite.) In this note, we shall characterize a Schwarz space by means of metrical dimensions of the same kind. For this purpose, we shall define more general approximate dimensions in an F-space E. An F-space E is called a Schwarz space if for every continuous semi-norm $p(x)$, there exists a continuous semi-norm $q(x)$ such that $U_q=\{x \in E, q(x) \leqq 1\}$ is totally bounded by the semi-norm $p(x)$. For subsets S and K of E, we shall define $N(K, \varepsilon S)$ as usual:

$$N(K, \varepsilon S)=\inf\left\{N : \bigcup_{k=1}^{N} (x_k+\varepsilon S) \supset K, x_k \in E\,;\, k=1,2,\cdots,N\right\}$$

for a real number $\varepsilon>0$.

An F-space E is a Schwarz space iff for every continuous semi-norm $p(x)$, there exists $q(x)$ such that $N(U_q, \varepsilon U_p)<+\infty$ for all $\varepsilon>0$.

Now, we shall consider two finite valued non-decreasing functions Φ, Ψ, each of which is defined on a sufficient large part of real numbers (i.e. $[\alpha, \infty)$ for some α), such that $\lim_{t\to\infty}\Phi(t)=\lim_{t\to\infty}\Psi(t)=+\infty$. Let $\{U_n\}_{n=1,2,\ldots}$ be any fundamental system of convex neighborhoods of zero in an F-space E. We shall define now another approximate dimension of E by Φ and Ψ as follows:

$$\rho_{\Phi,\Psi}(E)=\sup_k \inf_m \varlimsup_{\varepsilon\to 0} \frac{\Phi(N(U_m, \varepsilon U_k))}{\Psi(1/\varepsilon)}.$$

Since $\bigcap_{n=1}^{\infty} U_n=\{0\}$, it is easy to see that $\rho_{\Phi,\Psi}$ is determined uniquely by the topology of E (i.e. independent of the choice of $\{U_n\}_{n=1,2,\ldots}$).

Theorem. *An F-space E is a Schwarz space iff there exist non-decreasing finite valued functions Φ and Ψ with $\lim_{t\to\infty}\Phi(t)=\lim_{t\to\infty}\Psi(t)=+\infty$ such that $\rho_{\Phi,\Psi}(E)<+\infty$.*

Proof. It is easy to see that if $\rho_{\Phi,\Psi}(E)<+\infty$, then E is a Schwarz space. Suppose that E is a Schwarz space. Let $\{U_n\}_{n=1,2,\ldots}$ be a fundamental system of nbd. of zero in E which consists of convex sets. By assumption, we can find $k_n>n$ such that $N(U_{k_n}, \varepsilon U_n)<\infty$ for all $\varepsilon>0$. Let us define

$$f_n(1/\varepsilon)=N(U_{k_n}, \varepsilon U_n) \qquad \text{for} \quad 0<1/\varepsilon<\infty.$$

$f_n(1/\varepsilon)$ is a non-decreasing non-negative function with respect to $1/\varepsilon$ and greater than 1. Let m be a positive integer. For $\varepsilon>0$ with $m-1$