123. On the Existence of Solutions for System of Linear Partial Differential Equations with Constant Coefficients

By Yoshio SHIMADA
Sophia University

(Comm. by Kinjirô KUNUGI, M. J. A., June 12, 1971)

This paper is on the extension of a theorem by J. F. Treves (Lectures on linear partial differential equations with constant coefficients) for single linear partial differential equation to the case of system, which owes a great deal to the suggestions of Prof. Mitio Nagumo.

Let $\mathfrak{A}$ be a non-commutative algebra with unit over the complex numbers C, and let $[A,B]=AB-BA$ for all $A,B\in\mathfrak{A}$. Let $A_1, \cdots, A_n, B_1, \cdots, B_n$ be $2n$ elements of the algebra $\mathfrak{A}$, satisfying the following commutation relations:

(1) $[A_j, A_k]=[B_j, B_k]=0$ for $1\leq j, k\leq n$. $[A_j, B_k]=0$ for $j\neq k$.

(2) $[A_j, B_j]=I$ (unit element of $\mathfrak{A}$) for $1\leq j\leq n$.

Let $P(X)=P(X_1, \cdots, X_n)$ be a polynomial with complex coefficients, and p be a multi-index $(p_1, \cdots, p_n)$ of n integers ≥ 0, and let

$$P^{(p)}(X)=\left(\frac{\partial}{\partial X_1}\right)^{p_1}\cdots\left(\frac{\partial}{\partial X_n}\right)^{p_n}P(X_1, \cdots, X_n).$$

Lemma 1 (by lecture note of Treves). *Let $P(X)$, $Q(X)$ be the polynomials in n letters with complex coefficients, then*

$$Q(B)P(A)=\sum_p \frac{(-1)^{|p|}}{p\,!}P^{(p)}(A)Q^{(p)}(B),$$

where $A=(A_1, \cdots, A_n), B=(B_1, \cdots, B_n)$ satisfying the above commutation relations (1), (2), *and $|p|=p_1+\cdots+p_n, p\,!=p_1!\cdots p_n!$.*

Lemma 2. *Let $\boldsymbol{P}(X), \boldsymbol{Q}(X)$ be arbitrary square matrix of (m,m)-type such that its elements are polynomials in n letters with complex coefficients, then*

$${}^t(\boldsymbol{Q}(B)\boldsymbol{P}(A))=\sum_p \frac{(-1)^{|p|}}{p\,!}\,{}^t\boldsymbol{P}^{(p)}(A)\,{}^t\boldsymbol{Q}^{(p)}(B).$$

Proof. This lemma follows immediately by substituting the equality in Lemma 1.

Now, assume that $\mathfrak{A}$ is an algebra of linear mappings $\mathcal{D}\rightarrow\mathcal{D}$, where $\mathcal{D}$ is the linear space of infinitely differentiable complex valued functions on $\boldsymbol{R}^n$ with compact support. Let $\mathcal{L}_2=L_2\times\cdots\times L_2$, and inner product of $\mathcal{L}_2$ is defined by $(\boldsymbol{f}, \boldsymbol{g})_{\mathcal{L}_2}=\sum_{i=1}^m (f_i, g_i)_{L_2}$ for $\boldsymbol{f}=(f_1, \cdots, f_m)$,