7. *The Powers of an Operator of Class* C_ρ

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(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 12, 1972)

1. In a recent paper [4], M. J. Crabb gives the best bound $\sqrt{2}$ of the inequality proposed by C. A. Berger and J. G. Stampfli [2]:

$$\limsup_{n\to\infty} \|T^n x\| \leqq \sqrt{2}\, \|x\|,$$

for an operator T with $w(T)=1$, where $w(T)$ is the numerical radius of T given by

$$w(T)=\sup \{|(Tx, x)|\,;\, \|x\|=1\}.$$

Using his method, he proves also a generalization of a theorem of Berger-Stampfli [3] and Williams-Crimmins [6]. In the present note, we shall give a further generalization of Crabb's theorem in an elementary method basing on an idea of C. A. Berger and J. G. Stampfli.

2. Following after B. Sz. Nagy and C. Foiaş [5], let C_ρ be the set of all operators acting on a Hilbert space $\mathfrak{H}$ such that there exist a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace and a unitary operator U acting on $\mathfrak{K}$ satisfying

(1) $$T^m=\rho P U^m | \mathfrak{H} \qquad (m=1, 2, \cdots),$$

where P is the projection of $\mathfrak{K}$ onto $\mathfrak{H}$. (1) implies at once

(2) $$T^{*m}=\rho P U^{*m} | \mathfrak{H} \qquad (m=1, 2, \cdots).$$

It is well-known by [5] that

$$C_1=\{T \in \boldsymbol{B}(\mathfrak{H})\,;\, \|T\| \leqq 1\}$$

and

$$C_2=\{T \in \boldsymbol{B}(\mathfrak{H})\,;\, w(T) \leqq 1\}.$$

Therefore, the following theorem contains Crabb's theorem as a special case ($\rho=2$):

Theorem. *Suppose that* $T \in C_\rho (\rho \neq 1)$ *and that*

(3) $$\|T^n x\|=\rho$$

for some integer n *and a unit vector* x. *Then we have*

(i) $T^{n+1}x=0$,

(ii) $\|T^k x\|=\sqrt{\rho}$ *for* $k=1, 2, \cdots, n-1$,

(iii) $x, Tx, \cdots, T^n x$ *are mutually orthogonal*,

and

(iv) *The linear span* $\mathfrak{L}$ *of* $x, Tx, \cdots, T^n x$ *is a reducing subspace of* T.

3. **Proof.** Ad (i). Let T be as in (1). Then

$$\rho \|x\|=\|T^n x\|=\|\rho P U^n x\|=\rho \|P U^n x\|.$$