4. On a Nuclear Function Space on a Topological Space

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In this paper, we construct the nuclear function space on a topological space X with a measure μ such that $\mu(K)<\infty$ for every compact subset K of X.

The construction can, of course, be adopted for locally compact groups with a left or right Haar measure, and so the function spaces defined in the present paper include those due to Pietsh [1].

For nuclear spaces and the related notions, see [2]. Throughout this paper we denote by $C(X)$ the space of all continuous functions on a topological space X.

Definition. Let X be a topological space with a measure μ such that $\mu(K)<\infty$ for every compact subset K of X. A subspace $N(X)$ of $C(X)$ is said to have the property (A) if for every compact subset K there exist a compact subset $H\supset K$ and a positive real number ρ such that

$$\sup\{|\phi(x)|\,;\,x\in K\}\leqq\rho\left\{\int_H|\phi(x)|^2d\mu(x)\right\}^{1/2}$$

for every $\phi\in N(X)$.

Theorem 1. *Let X be a topological space with a measure μ such that $\mu(K)<\infty$ for every compact subset K of X. Let $N(X)$ be a subspace of $C(X)$ with the property (A).*

Define the topology of $N(X)$ by the system of seminorms

$$\|\phi\|_K=\left\{\int_K|\phi(x)|^2d\mu(x)\right\}^{1/2},\qquad K\in\mathfrak{S}$$

where $\mathfrak{S}$ is the family of all compact subsets of X. Then the locally convex space $N(X)$ is a nuclear space.

Proof. Let K be a member of $\mathfrak{S}$ and let $\wedge$ be a canonical mapping of $N(X)$ into $N(X)/\{\phi\,;\,\|\phi\|_K=0\}$. We denote by H_K the completion of the quotient space $N(X)/\{\phi\,;\,\|\phi\|_K=0\}$ with respect to the quotient norm. Then it is clear that H_K is a Hilbert space.

In order to show that $N(X)$ is a nuclear space, it suffices to prove that for any compact subset K there exists a compact subset $T\supset K$ such that the canonical mapping $H_T\to H_K$ is of Hilbert-Schmidt type. By the assumption, we have the inequality

$$\sup\{|\phi(x)|\,;\,x\in K\}\leqq\rho\left\{\int_T|\phi(x)|^2d\mu(x)\right\}^{1/2}$$

for every $\phi\in N(X)$.