3. On Integral Inequalities Related with a Certain Nonlinear Differential Equation

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(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 12, 1972)

As is shown in [3], the following nonlinear differential equation:

$$(1) \qquad nh(1-h^2)\frac{d^2h}{dt^2}+\left(\frac{dh}{dt}\right)^2+(1-h^2)(nh^2-1)=0,$$

where n is any integer $\geqq 2$, is the equation for the support function $h(t)$ of a plane curve in the unit disk: $u^2+v^2<1$, with respect to the tangent direction angle t, which is related with a minimal hypersurface in the $(n+1)$-dimensional unit sphere. Any solution $h(t)$ of (1) such that $h^2+\left(\frac{dh}{dt}\right)^2<1$ is periodic and its period T is given by the improper integral:

$$(2) \qquad T(C)=2\int_{a_0}^{a_1}\frac{dh}{\sqrt{1-h^2-C\left(\frac{1}{h^2}-1\right)^{1/n}}},$$

where

$$C=(a_0^2)^{1/n}(1-a_0^2)^{1-(1/n)}=(a_1^2)^{1/n}(1-a_1^2)^{1-(1/n)}$$
$$\left(0<a_0<\frac{1}{\sqrt{n}}<a_1\right)$$

is the integral constant of (1). Regarding the function $T(C)$, $0<C<A=(1/n)^{1/n}(1-(1/n))^{1-(1/n)}$, the following is known in [3]:

(i) *$T(C)$ is differentiable and $T(C)>\pi$,*

(ii) $\lim_{C\to 0} T(C)=\pi$ *and* $\lim_{C\to A} T(C)=\sqrt{2}\,\pi$.

Putting $h^2=x$, $a_0^2=x_0$, $a_1^2=x_1$ and $1/n=\alpha$, (2) can be written as

$$(3) \qquad T(C)=\int_{x_0}^{x_1}\frac{dx}{\sqrt{x(1-x)-C\psi(1-x)}},$$

where

$$(4) \qquad \psi(x)=x^{\alpha}(1-x)^{1-\alpha} \qquad \text{on } 0<x<1$$

and

$$(5) \qquad C=\psi(x_0)=\psi(x_1), \qquad 0<x_0<\alpha<x_1<1,$$

$$(6) \qquad 0<C<A=\psi(\alpha).$$

Now, suppose that α is any real number such that

$$(7) \qquad 0<\alpha\leqq 1/2$$

and consider as the function $T(C)$ is defined by the right hand side of (3) on the interval (6). Then, we have

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*) Dedicated to Professor Yoshie Katsurada on her 60th birth day.