27. A Remark on the Approximate Spectra of Operators

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1. In the present note, several equivalent conditions on the approximate spectrum of an operators will be discussed in § 2. The joint approximate spectrum introduced by Bunce [5] is also discussed in § 4. In § 3, an algebraic proof of Wintner-Hildebrandt-Orland's theorem is given.

2. The equivalence of several definitions on an approximate propervalue is unified in the following theorem:

Theorem 1. *For an operator T on a Hilbert space $\mathfrak{H}$, the following conditions are equivalent:*

(i) *For any $\varepsilon>0$, there is a vector $x \in \mathfrak{H}$ with $\|x\|=1$ and*

(1) $$\|Tx-\lambda x\|<\varepsilon,$$

(ii) *There is a sequence of operators S_n with $\|S_n\|=1$ and*

(2) $$\|(T-\lambda)S_n\|\to 0 \qquad (n\to\infty),$$

(iii) *Let $\mathfrak{B}(\mathfrak{H})$ be the algebra of all operators, then*

(3) $$\mathfrak{B}(\mathfrak{H})(T-\lambda)\neq\mathfrak{B}(\mathfrak{H}),$$

(iv) *There is no $\varepsilon>0$ such that*

(4) $$(T-\lambda)^*(T-\lambda)\geqq\varepsilon.$$

Historically, (i) is the original definition of Halmos [7; p. 51], (ii) is due to Berberian [1; VII, § 3, Ex. 10], (iii) is introduced very recently by Bunce [4] and (iv) is pointed out by Berberian [2].

If λ satisfies one of the above conditions, λ will be called an *approximate propervalue* of T. The set $\pi(T)$ of all approximate propervalues of T is called the *approximate spectrum* of T.

(i) implies (ii): This is already contained in [1]. Suppose
$$\|Tx_n-\lambda x_n\|\to 0 \qquad (n\to\infty)$$
for $\|x_n\|=1$. If $S_n=x_n\otimes x_n$ in the sense of Schatten [11], i.e.
$$(y\otimes z)x=(x\,|\,z)y,$$
then S_n is a one-dimensional projection, so that
$$\|S_n\|=1, \quad \|(T-\lambda)S_n\|\to 0 \quad (n\to\infty).$$

(ii) implies (iii): $T-\lambda$ is a right generalized divisor of zero [10; p. 27]; hence $\mathfrak{B}(\mathfrak{H})(T-\lambda)$ consists of generalized divisors of zero which implies (iii).

(iii) implies (iv): If not,

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