14. A Note on Schütte's Interpolation Theorem

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(Comm. by Kunihiko KODAIRA, M. J. A., Feb. 12, 1972)

In this note, we shall add some remarks on Schütte's interpolation theorem in the intuitionistic predicate logic (cf. Schütte [3]), one of which give an affirmative solution of one of open problems in Gabbay [1].

Schütte's interpolation theorem. *If $A \supset B$ is provable in the intuitionistic predicate logic, then there is a formula C satisfying the following* (1) *and* (2):

(1) *$A \supset C$ and $C \supset B$ are provable in this logic.*

(2) *Every predicate symbol in C occurs both in A and in B.*

We add the following fact to this theorem:

Theorem. *In Schütte's theorem above, if A and B are built up using $\neg$ (negation), $\wedge$ (conjunction) and $\forall$ (universal quantification) only, then we can take such a C which satisfies* (1), (2) *and an added condition* (3):

(3) *Every free variable in C occurs both in A and in B.*

Remark 1. *The proposition obtained from the above theorem by omitting* (3) *is an affirmative solution of one of open problems in* [1].

Remark 2. *In Schütte's theorem, we can easily add the condition* (3) *to C, but in our theorem this is not trivial because we can not apply $\exists$ (existential quantifier) to C.*

Let LJ be the intuitionistic predicate logic formulated by Gentzen in [2]. For the sake of simplicity we assume that a sequent in LJ is of the form $\Gamma \to \Theta$, where Γ and Θ are finite sets of formulas in LJ and Θ has at most one formula, although we shall write $A, \Gamma \to B$ instead of $\{A\} \cup \Gamma \to \{B\}$. Furthermore we assume that LJ has two propositional constants $\top$ (truth), $\perp$ (false) and two added axiom sequents $\to \top$ and $\perp \to$.

Lemma 1. *Let $\Gamma \to \Theta$ be a sequent in LJ and (Γ_1, Γ_2) be an ordered partition of Γ. If $\vdash_{LJ} \Gamma \to \Theta$, then there is a formula C such that*

(4) *$\vdash_{LJ} \Gamma_1 \to C$ and $\vdash_{LJ} C, \Gamma_2 \to \Theta$.*

(5) *Every predicate symbol in C occurs both in Γ_1 and $\Gamma_2 \cup \Theta$. Furthermore if every formula in $\Gamma \cup \Theta$ is built up using $\neg, \wedge, \forall$ only, then C is also such a formula.*

Proof. We use the induction on a cut-free derivation $\mathcal{D}$ of $\Gamma \to \Theta$. We only treat the case that the last rule of $\mathcal{D}$ is $(\neg \to)$ or $(\to \forall)$.

Case 1. The last rule of $\mathcal{D}$ is $(\neg \to)$. Then $\mathcal{D}$ has the form