50. On Normal Approximate Spectrum

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1. Introduction. Bunce [1] established a kind of the reciprocity among the characters of singly generated C^*-algebras and the approximate spectra of the generators of certain classes. Kasahara and Takai [5] introduced the notion of the normal approximate spectra and gave new proofs of the main theorems of Bunce [1]. However, Kasahara and Takai remain the reciprocity of characters and spectra.

In the present note, we shall complete the reciprocity of Bunce with the use of the idea due to Kasahara and Takai in § 2. We shall also discuss the joint normal approximate spectra in § 3. In § 4, we shall show that a theorem of Coburn [3] is given an elementary proof based on the reciprocity obtained in § 2.

2. Reciprocity. Let A be a (bounded linear) operator on a Hilbert space $\mathfrak{H}$. Let $\pi(A)$ be the approximate spectrum of A, cf. [4]. Following after the definition of Kasahara and Takai [5], a complex number λ is a *normal approximate propervalue* of A if there exists a sequence $\{x_n\}$ of unit vectors such that

$$\|(A-\lambda)x_n\|\to 0 \quad \text{and} \quad \|(A-\lambda)^*x_n\|\to 0 \quad (n\to\infty).$$

The set $\pi_n(A)$ of all normal approximate propervalues is called the *normal approximate spectrum* of A.

Let us begin to prove the following lemma which is essentially due to Mr. H. Takai:

Lemma 1. *$\lambda\in\pi_n(A)$ if and only if $(A-\lambda)^*(A-\lambda)+(A-\lambda)(A-\lambda)^*$ is not strictly positive,* i.e. *there is no $\varepsilon>0$ such that*

$$(A-\lambda)^*(A-\lambda)+(A-\lambda)(A-\lambda)^*\geqq\varepsilon. \tag{1}$$

Proof. By the equality

$$(((A-\lambda)^*(A-\lambda)+(A-\lambda)(A-\lambda)^*)x\,|\,x) = \|(A-\lambda)x\|^2+\|(A-\lambda)^*x\|^2, \tag{2}$$

$\lambda\in\pi_n(A)$ implies there is a sequence $\{x_n\}$ of unit vectors such that the both terms of the right-hand side of (2) tend to 0, so that (1) is not satisfied.

Conversely, if (1) is not satisfied, then 0 is contained in the closure $\bar{W}((A-\lambda)^*(A-\lambda)+(A-\lambda)(A-\lambda)^*)$ of the numerical range of the operator described in the left-hand side of (1), so that $\lambda\in\pi_n(A)$.

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