109. Structure of Left QF-3 Rings

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The purpose of this note is to establish a structure theorem for left QF-3 rings, an analogue to one for QF-3 algebras by Morita [14], introducing a new notion of left QF-3 rings.

It turns out that not only faithful projective-injective modules but also dominant modules play a vital role in the structure theory of left (-right) QF-3 rings.

Throughout this note, rings R and S will have identity and modules will be unital. ${}_SX$ will signify the fact that X is a left S-module. We adopt the notational convention of writing module-homomorphism on the side opposite the scalars.

Definition (Kato [10]). A module P_R is called dominant if P_R is faithful finitely generated projective and ${}_SP$ is lower distinguished[1] with $S=\mathrm{End}(P_R)$.

The following definition of left QF-3 rings finds no mention in the literature.

Definition. A ring R will be called left QF-3 if R contains idempotents e and f such that Re is a faithful injective left ideal and fR is a dominant right ideal.

Lemma 1[2]. *If e and f are idempotents of R such that ${}_RRe$ is injective and fR_R is faithful, then*

(1) $Re=\mathrm{Hom}({}_{fRf}fR, {}_{fRf}fRe)$, so $eRe=\mathrm{End}({}_{fRf}fRe)$.

(2) ${}_{fRf}fRe$ *is injective.*

Proof. This is Proposition 2.1 of Tachikawa [25].

Lemma 2. *The double centralizer of any faithful torsionless right R-module is a left quotient*[3] *ring of R.*

Proof. See Colby and Rutter [3, 4], Tachikawa [25], Faith [5], and Kato [11].

Lemma 3. *Let ${}_SV$ be a cogenerator and $T=\mathrm{End}({}_SV)$. Then ${}_SV$ is linearly compact if and only if V_T is injective; then a module ${}_SU$ is linearly compact if and only if ${}_SU$ is V-reflexive.*

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1) ${}_SP$ is lower distinguished if ${}_SP$ contains a copy of each simple module. Cf. Azumaya [1].

2) Cf. Kato [13].

3) Q is a (the maximal) left quotient ring of R if Q is a ring extension of R and ${}_RQ$ is a (the maximal) rational extension of ${}_RR$. Cf. Findlay and Lambek [6].