144. On the Structure of Single Linear Pseudo-Differential Equations

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The purpose of this note is to determine the structure of some class of *single* (linear) pseudo-differential equations by the aid of "quantized" contact transformations. (Cf. Egorov [1], Hörmander [4] and Sato, Kawai and Kashiwara [8].) It extends a result in § 2 of Chapter III of Sato, Kawai and Kashiwara [8] under the assumption of *single* equations.

Our main result is the following.

Theorem. *Let $P(x, D)$ be a pseudo-differential operator defined in a complex neighborhood U of $x_0^*=(x_0, \sqrt{-1}\,\eta_0) \in \sqrt{-1}\, S^*M$, where M is an n-dimensional real analytic manifold. Denote its principal symbol by $P_m(x, \eta)$. Assume that $P(x, D)$ satisfies conditions* (1) *and* (2) *below.*

Then the equation $P(x, D)u=0$ is micro-locally equivalent to one of the Mizohata equations

$$\mathfrak{M}_{k,l}^{\pm}: \left(\frac{\partial}{\partial x_1} \pm \sqrt{-1}\, x_1^k \frac{\partial}{\partial x_2}\right)^l u=0$$

considered near $(0; \sqrt{-1}\,(0, 1, 0, \cdots, 0))$ for some positive integers k and l.

(1) *$V=\{(z, \zeta) \in U \mid P_m(z, \zeta)=0\}$ is a non-singular manifold.* (*Note that its defining ideal is not necessarily reduced.*)

(2) *There exist holomorphic functions $f_1(z, \zeta)$ and $f_2(z, \zeta)$ homogeneous in ζ such that $f_1=f_2=0$ on $V \cap \overline{V}$, $\overline{V}$ denoting the complex conjugate of V, and that their poisson bracket $\{f_1, f_2\}$ never vanishes.*

Proof. We denote by $Q(z, \zeta)$ a generator of the reduced defining ideal of V, i.e. $P_m=Q^l$. Then condition (2) assures that $d_{(z,\zeta)}Q(z, \zeta)$ and the canonical 1-form $\omega=\sum_{j=1}^n \zeta_j dz_j$ are linearly independent in a neighborhood of x_0^*. Hence by a suitable contact transformation we may assume without loss of generality that $Q(z, \zeta)$ has the form

$$\zeta_1+\sqrt{-1}\,\varphi(z, \zeta), \tag{3}$$

where $\varphi(z, \zeta)$ is real-valued on S^*M and that $V \cap \overline{V}=\{(x, \zeta) \mid z_1=0, \zeta_1=0\}$ (cf. Lemma 2.3.3 in Chapter III of Sato, Kawai and Kashiwara [8]). Then clearly $V \cap \overline{V}=\{\zeta_1=\varphi(z, \zeta)=0\}$. We can assume without loss of generality $(x_0, \eta_0)=(0; (0, 1, 0, \cdots, 0))$. Therefore we can find an integer k so that $\varphi_0(z, \zeta')=\varphi(z, \zeta)|_{\zeta_1=0}$ has the form $\pm z_1^k \chi(z, \zeta')$ where χ never