# 38. A von Neumann Algebra Continuous over a von Neumann Subalgebra

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**1.** In [3], we have introduced a generalized notion of abelian projections of von Neumann algebras and proved that some elementary properties of abelian projections are preserved under the generalization.

Using this concept, in this note, we shall introduce a notion that a von Neumann algebra is continuous over a von Neumann subalgebra, and show some properties of such a von Neumann algebra in § 2.

In § 3, we shall prove that a von Neumann algebra $\mathcal{A}$ continuous over a von Neumann subalgebra $\mathcal{B}$ has an useful property relative to an expectation of $\mathcal{A}$ onto $\mathcal{B}$. In [2], H. Choda has introduced the notion of Maharam subalgebras of von Neumann algebras motivated by Maharam's lemma. On the analogy of this definition, we shall introduce a notion of strong Maharam subalgebras of von Neumann algebras and prove that a von Neumann subalgebra $\mathcal{B}$ of a von Neumann algebra $\mathcal{A}$ contained in the center is a strong Maharam subalgebra of $\mathcal{A}$ if $\mathcal{A}$ is continuous over $\mathcal{B}$.

We shall use the terminology due to Dixmier [4] throughout this note without further explanations.

**2.** In the sequel, let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{B}$ a von Neumann subalgebra of $\mathcal{A}$. Denote by $\mathcal{B}^c=\mathcal{A}\cap\mathcal{B}'$ the relative commutant of $\mathcal{B}$ in $\mathcal{A}$ and $\mathcal{B}^P$ the set of all projections in $\mathcal{B}$.

The following definition is introduced in [3] as a generalization of the notion of abelian projections:

**Definition 1.** A projection $E\in\mathcal{A}$ is called to be *abelian over* $\mathcal{B}$ if $E\in\mathcal{B}^c$ and, for every projection $P\in\mathcal{A}$ such that $P\leqq E$, there exists a projection $Q\in\mathcal{B}$ such that $P=QE$.

The following lemma gives an alternative algebraic definition of abelian projections over $\mathcal{B}$:

**Lemma 2.** *$E\in\mathcal{A}^P$ is abelian over $\mathcal{B}$ if and only if $E\in\mathcal{B}^c$ and $E\mathcal{A}E=\mathcal{B}E$.*

**Proof.** The "only if" part is obvious. Conversely, let $E$ be a projection in $\mathcal{B}^c$ such that $E\mathcal{A}E=\mathcal{B}E$ and $\tilde{E}$ the $\mathcal{B}\cap\mathcal{B}'$-support of $E$, that is,

$$\tilde{E}=\inf\{F\in(\mathcal{B}\cap\mathcal{B}')^P\,;\,F\geqq E\},$$