37. On the Logarithm of Closed Linear Operators

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For a non-negative operator A in a Banach space X, Nollau [3] gave a definition of its logarithm $\log A$. In this note, we present another definition of $\log A$. Formally our definition is based on the relation

$$\log A = \log A(\mu+A)^{-1} - \log(\mu+A)^{-1}, \qquad \mu>0.$$

It is important here that $\log(\mu+A)^{-1}$ (resp. $\log A(\mu+A)^{-1}$) is to be defined as the infinitesimal generator of a holomorphic semi-group $(\mu+A)^{-\alpha}$, $\alpha\geqq 0$, (resp. $A^{\alpha}(\mu+A)^{-\alpha}$) under suitable conditions on A. Using this relation, we derive several formal properties of $\log A$, of which some seem to be new. By means of these properties, we finally give another proof of one of Nollau's representation formulas for $\log A$. The original proof was done through Dunford's integral and Nollau relied on this formula for the derivation of formal properties of $\log A$.

1. Definition and formal properties. We only consider a densely ranged and densely defined non-negative operator A in a Banach space X. Namely, all positive reals are contained in the resolvent set $\mathbf{P}(-A)$ of $-A$;

(1.1) $$\|r(r+A)^{-1}\|\leqq M, \qquad r>0;$$

(1.2) $$\overline{D(A)}=X;$$

(1.3) $$\overline{R(A)}=X.$$

Here $D(T)$, $R(T)$ stand for the domain and the range of an operator T, respectively. $\overline{Y}$ is the closure of the set Y in X.

For A with (1.1), (1.2), (1.3), the following assertion is well-known (Komatsu [1, 2], cf. Yosida [4]).

Proposition 1.1. *For any positive* μ, $\{(\mu+A)^{-\alpha};\alpha\geqq 0\}$, $\{A^{\alpha}(\mu+A)^{-\alpha};\alpha\geqq 0\}$ *are strongly continuous semi-groups of bounded linear operators. Both semi-groups are analytically continued to the half plane* $\mathrm{Re}\,\alpha>0$.

We also note the following relation (cf. Komatsu [2]):

(1.4) $$A^{\alpha}(\mu+A)^{-\alpha}=\mu^{-\alpha}(A^{-1}+\mu^{-1})^{-\alpha}.$$

We denote by $\Lambda^{+}(\mu;A)$ (resp. $\Lambda^{-}(\mu;A)$) the infinitesimal generator of $(\mu+A)^{-\alpha}$ (resp. $A^{\alpha}(\mu+A)^{-\alpha}$). We set $D^{\pm}(\mu;A)=D(\Lambda^{\pm}(\mu;A))$. We sometimes write $\Lambda^{\pm}(\mu)$, $D^{\pm}(\mu)$ instead of $\Lambda^{\pm}(\mu;A)$, $D^{\pm}(\mu;A)$.

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