50. *Cauchy Problem for Degenerate Parabolic Equations*

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1. Introduction. We consider the Cauchy problem for the equation

$$\begin{aligned}(1.1)\qquad &\partial_t u-\sum_{j,k=1}^{n}\partial_{x_j}(a_{jk}(x,t)\partial_{x_k}u)-\sum_{j=1}^{n}b_j(x,t)\partial_{x_j}u-c(x,t)u\\ &=\partial_t u-Au=f,\end{aligned}$$

(x,t) in $\boldsymbol{R}^n\times[0,\infty)$ with the initial-value

$$(1.2)\qquad u(x,0)=u_0(x),$$

where $a_{jk}(x,t)$, $b_j(x,t)$, $c(x,t)$ are real-valued smooth functions. We assume that $(a_{jk})_{1\le j\le n,1\le k\le n}$ is symmetric and satisfies the condition: for any $(x,t)\in\boldsymbol{R}^n\times[0,\infty)$

$$(1.3)\qquad \sum_{j,k=1}^{n}a_{jk}(x,t)\xi_j\xi_k\ge 0 \qquad \text{for all } \xi\in\boldsymbol{R}^n.$$

O. A. Oleĭnik has treated this problem (see [3] and [4]). Her method consists of the following procedure (elliptic regularization): Instead of (1.1), the following equations (depending on a positive parameter ε) in $G=\boldsymbol{R}^n\times[0,T]$

$$(1.4)\qquad \partial_t u-\varepsilon\Delta u-Au=f$$

are considered. Let u_ε be the solution of (1.4) with the given initial-value $u_0(x)\in L^2(\boldsymbol{R}^n)$ and $f(x,t)\in L^2(G)$. Then it is shown that $\{u_\varepsilon(x,t)\}$ is bounded in $L^2(G)$. Then a weak limit of them, as $\varepsilon\to+0$, gives the desired solution $u(x,t)\in L^2(G)$. The uniqueness of the solution is proved. She also proved the smoothness of u, assuming the smoothness of u_0 and f.

Contrary to the above point of view, we regard (1.1) as evolution equation. More precisely, we want to show the existence of the unique solution $u(x,t)\in\mathcal{E}_t^0(L^2)\cap\mathcal{E}_t^1(\mathcal{D}'^2_{L^2})$ of (1.1)–(1.2) for any $f(x,t)\in\mathcal{E}_t^0(L^2)$ and any initial-value $u_0(x)\in L^2$.*)

Our approach is based on the semi-group theory. Instead of elliptic regularization, we use Friedrichs' mollifier. Its property (see

*) Throughout this paper, we use the following notation: $x=(x_1,\cdots,x_n)$. $\partial_t=\partial/\partial t$, $\partial_j=\partial x_j=\partial/\partial x_j$, $\partial_x^\nu=\partial_1^{\nu_1}\cdots\partial_n^{\nu_n}$, where $\nu=(\nu_1,\cdots,\nu_n)$. $L^2=L^2(\boldsymbol{R}^n)$. $u(x)\in\mathcal{D}_{L^2}^m$ means that its derivatives (in the sense of distribution) $\partial_x^\nu u$ up to order m belong to L^2. $\mathcal{D}'^m_{L^2}$ is the dual space of $\mathcal{D}^m_{L^2}$ and sometimes we denote it by $\mathcal{D}^{-m}_{L^2}$. $\varphi(x)\in\mathcal{B}^m$ means that its derivatives $\partial_x^\nu\varphi$ up to order m are continuous and bounded in $\boldsymbol{R}^n$. $f(t)\in\mathcal{E}_t^k(\mathcal{D}^m_{L^2}$ (or $\mathcal{B}^m$)) means that $t\to f(t)\in\mathcal{D}^m_{L^2}$ (or $\mathcal{B}^m$) is continuously differentiable up to order k.