95. On Strongly Regular Rings

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A ring R is called *strongly regular* if for every element $a \in R$ there exists an element $x \in R$ such that $a=a^2x$. As is well-known, R is strongly regular if and only if one of the following equivalent conditions is satisfied:

(A) For every element $a \in R$ there holds $a \in aR$ and there exists a central idempotent e such that $aR=eR$.

(B) R is a regular ring without nonzero nilpotent elements. Obviously, the notion "strongly regular" is right-left symmetric. Next, a ring R is called a *right* [*left*] *duo ring* if every right [left] ideal of R is an ideal. Finally, a ring R is called a *right* [*left*] *V-ring* if $R^2=R$ and every right [left] ideal of R is an intersection of maximal right [left] ideals of R.

It is the purpose of this note to prove the following that contains [2; Theorem 2], [5; Theorem] and [7; Theorem 3 and Corollary 1]:

Theorem. *The following conditions are equivalent*:

(1) *R is strongly regular.*

(2) *R is a regular ring and is a subdirect sum of division rings.*

(3) *$\mathfrak{l} \cap \mathfrak{r} = \mathfrak{l}\mathfrak{r}$ for every left ideal $\mathfrak{l}$ and every right ideal $\mathfrak{r}$ of R.*

(4) *R contains no nonzero nilpotent elements and $R/\mathfrak{p}$ is regular for every prime ideal $\mathfrak{p} \subsetneqq R$.*

(5) *R is a regular, right duo ring.*

(6) *$\mathfrak{r} \cap \mathfrak{r}' = \mathfrak{r}\mathfrak{r}'$ for each right ideals $\mathfrak{r}, \mathfrak{r}'$ of R.*

(7) *R is a right duo ring such that every ideal is idempotent.*

(8) *R is a right duo, right V-ring.*

(9) *R contains no nonzero nilpotent elements and every completely prime ideal $\subsetneqq R$ is a maximal right ideal.*

(5′)–(9′). *The left-right analogues of* (5)–(9).

In the proof of our theorem, we shall use several familiar results, which are summarized in the next lemma.

Lemma. *Let R be a ring without nonzero nilpotent elements, and let a, b be elements of R.*

(a) *If $ab=0$ then $ba=0$, and so the right annihilator $r(a)$ coincides with the left one $l(a)$.*

(b) *If a is nonzero then $R/r(a)$ contains no nonzero nilpotent elements and the residue class $\bar{a}$ of a* mod *$r(a)$ is a non-zero-divisor.*