83. *The Solution Spaces of Non-Linear Partial Differential Equations of Elliptic Type on Compact Manifolds*

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§ 1. In this paper, we shall deal with some geometric properties concerning with the *solution space* of non-linear partial differential equations of elliptic type defined on compact manifolds, by using a unified method, namely that of the "linearization" of the non-linear operators.

Throughout the present paper, let M denote a compact C^∞-manifold of dimension n, and $C^\infty(M)$ the linear space of C^∞-functions on M with the C^∞-topology. Further, let m be an arbitrary non-negative integer. Then we define a (non-linear) differential operator L of order m on M as a mapping:

$$L: C^\infty(M) \to C^\infty(M),$$

which can be expressed, locally, in terms of coordinates, as a C^∞-function in the partial derivatives of order $\leqq m$. To state more precisely, let $x_1, \cdots, x_n$ denote the local coordinates of M with the coordinate domain U, then the operator: $C^\infty(U) \to C^\infty(U)$ induced by L has the form $L(u) = F(x, D_x^\alpha u)$, where $F(x, y^{(\alpha)})$ is an element of $C^\infty(U \times \boldsymbol{R}^N)$ (N denotes the number of multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $|\alpha| = \sum \alpha_i \leqq m$), and D_x^α denotes the partial derivative $\partial^{\alpha_1}/\partial x_1^{\alpha_1} \cdots \partial^{\alpha_n}/\partial x_n^{\alpha_n}$.

In our case, the *linearization* (the Gateaux derivative) of L at $f \in C^\infty(M)$ is given by

$$d_f L(u) = \lim_{h \to 0} \frac{L(f+hu) - L(f)}{h}.$$

If L has the local expression as above, it can be expressed by

$$d_f L(u) = \sum_{|\alpha| \leqq m} \frac{\partial F(x, y^{(\alpha)})}{\partial y^{(\alpha)}}\bigg|_{y^{(\alpha)} = D_x^\alpha f} D^\alpha u.$$

Hence, $d_f L$ is a *linear* differential operator with C^∞-coefficients of order m.

The operator L will be called an *elliptic operator* (of order m), if, for each $f \in C^\infty(M)$ and for each local parameter, the highest order term of $d_f L$

$$\sum_{|\alpha| \leqq m} \frac{\partial F(x, y^{(\alpha)})}{\partial y^{(\alpha)}}\bigg|_{y^{(\alpha)} = D_x^\alpha f} \xi^\alpha$$