115. *Characterizations of Compactness and Countable Compactness*

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It is known that if a topological space Y is compact, then the following condition is satisfied.

($*$) For every topological space X, each mapping of X into Y with closed graph is continuous.

The purpose of this note is to show that this condition characterizes compact spaces among T_1 spaces by proving somewhat strengthened result. A similar characterization of countably compact spaces is also stated.

Recall that a net in a set X is an ordered pair $(f,(D,\leqslant))$ of a directed set $(D,\leqslant)$ and a mapping f of D into X. If a is an element of a directed set $(D,\leqslant)$, we denote by $D(a)$ the set of all $x\in D$ with $a\leqslant x$.

Let $\mathcal{S}$ be a class of topological spaces containing the class of Hausdorff completely normal and fully normal spaces. Thus for example $\mathcal{S}$ may be the class of Hausdorff completely regular spaces or that of paracompact spaces. We have the following

Theorem 1. *A T_1 topological space Y is compact if and only if for every topological space X belonging to $\mathcal{S}$, each mapping of X into Y with closed graph is continuous.*

Proof. Only the proof of the "if" part is needed. Suppose that Y is not compact. Then there is a net $(f,(D,\leqslant))$ in Y which has no adherent point. Let $\infty\notin D$, and let $X=D\cup\{\infty\}$. It is easy to see that the family $\mathcal{P}(D)\cup\{D(x)\cup\{\infty\}\mid x\in D\}$ is a base for a topology τ on X, where $\mathcal{P}(D)$ denotes the power set of D.

To prove that τ is Hausdorff, it suffices to show that for every $x\in D$, there is an element $y\in D\backslash\{x\}$ with $x\leqslant y$, since this implies $\{x\}\cap(D(y)\cup\{\infty\})=\emptyset$. To this end suppose the contrary: there is an $x\in D$ such that $x\leqslant y$ does not hold for any $y\in D\backslash\{x\}$. If $y\in D$, then we have $x\leqslant z$ and $y\leqslant z$ for some $z\in D$, and consequently $z=x$ and $y\leqslant x$. Therefore we have $y\leqslant x$ for all $y\in D$, which yields however a contradiction that $f(x)$ is an adherent point of the net $(f,(D,\leqslant))$.

Let us proceed to prove that (X,τ) is completely normal. Let A and B be separated subsets of X, i.e., $\overline{A}\cap B=A\cap\overline{B}=\emptyset$. If $\infty\notin\overline{A}$, then $\overline{A}$ and $\overline{A}^c=X\backslash\overline{A}$ are open disjoint and $B\subset\overline{A}^c$. If $\infty\notin B$, then B