133. *Bounded Variation Property of a Measure*

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1. Introduction. For an integral structure $\Gamma=(\Lambda;\mathcal{S},\mathcal{G},Q)$ defined in [3], we shall discuss in this paper a certain type of bounded variation property of a pre-measure $\mu\in Q$. Through the discussion, some properties of the 'indefinite integral' $\sigma(\cdot,f,\mu)$, where σ is an integral with respect to Γ, and a theorem similar to Lebesgue's bounded convergence theorem will be obtained.

2. Bounded variation property.

Assumption 1. *M is a set and $\mathcal{S}$ is a ring of subsets of M. G is a topological additive group and μ is a G-valued pre-measure on $\mathcal{S}$.*

Let us denote by $\mathcal{V}$ the system of neighbourhoods of $0\in G$.

The pre-measure μ is *locally s-bounded* if, for any $X\in\mathcal{S}$ and $X_i\in\mathcal{S}$, $i=1,2,\cdots$, such that $X_jX_k=0$ $(j\neq k)$, and for any $V\in\mathcal{V}$, there exists a positive integer n such that $\mu(XX_i)\in V$ for any $i\geqq n$.

Proposition 1. *If $\mathcal{S}$ is a pseudo-σ-ring and μ is a measure, then μ is locally s-bounded.*

Proof. Let X and X_i, $i=1,2,\cdots$, be elements of $\mathcal{S}$ such that $X_jX_k=0$ $(j\neq k)$ and V an element of $\mathcal{V}$. Since $\mathcal{S}$ is a pseudo-σ-ring, $Y_n=\bigcup_{i=n}^{\infty}XX_i$ is an element of $\mathcal{S}$ for each $n=1,2,\cdots$. Since μ is a measure, it follows from $Y_n\downarrow 0$ $(n\to\infty)$ that $\mu(Y_n)\to 0$ $(n\to\infty)$. Hence, for an element V_0 of $\mathcal{V}$ such that $V_0-V_0\subset V$, we have a positive integer n such that $\mu(Y_i)\in V_0$ for any $i\geqq n$. For this n and for any $i\geqq n$, we have $\mu(XX_i)=\mu(Y_i-Y_{i+1})=\mu(Y_i)-\mu(Y_{i+1})\in V_0-V_0\subset V$, which proves the proposition.

For an element V of $\mathcal{V}$, an element X of $\mathcal{S}$ is *of V-variation* if $\mu(XY)\in V$ for any $Y\in\mathcal{S}$.

Then the following is easily seen:

Proposition 2. *If an element X of $\mathcal{S}$ is of V-variation with $V\in\mathcal{V}$, then XY is of V-variation for any $Y\in\mathcal{S}$.*

Proposition 3. *Suppose that μ is a locally s-bounded measure and $X_i\downarrow 0$ $(i\to\infty)$ for $X_i\in\mathcal{S}$, $i=1,2,\cdots$. Then for any $V\in\mathcal{V}$ there exists a positive integer n such that X_n is of V-variation.*

Proof. Let us assume that no X_i is of V-variation. Let V_0 be an element of $\mathcal{V}$ such that $2V_0\subset V$. Put $i_0=1$ and assume that a positive integer i_{n-1} is defined. Then we have an element $Y_{i_{n-1}}$ of $\mathcal{S}$ such that $Y_{i_{n-1}}\subset X_{i_{n-1}}$ and $\mu(Y_{i_{n-1}})\notin V$. Since $Y_{i_{n-1}}X_j\downarrow 0$ $(j\to\infty)$ implies $\mu(Y_{i_{n-1}}X_j)$